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# A QUASIANALYTICITY CRITERION

MATHEMATICS

1970

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**Abstract**

**Full Text**

UDC 513.88 : 513.83

*MATHEMATICS*

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## A QUASIANALYTICITY CRITERION FOR ABSTRACT OPERATORS

*(Presented by Academician V. I. Smirnov, January 7, 1969)*

The present note is an attempt to extend Carleman's quasianalyticity criterion<sup>(1,2)</sup> to linear operators in a Banach space. We shall be concerned with conditions under which from inequalities of the form

$$\|A^n x\| \leq m_n \|x\| \quad (n = 0, 1, 2, \dots) \quad (1)$$

it follows that  $x = 0$ . Classical quasianalyticity corresponds to the case when  $A = \frac{1}{i} \frac{d}{ds}$  in the space  $C[0, 1]$  with the boundary condition  $x(0) = 0$ . For more general differential operators the problem of quasianalyticity was studied in<sup>(3-5)</sup>.

The differentiation operator mentioned above is unbounded and has no spectrum. These properties should be retained in the general formulation of the problem: if on some nonzero invariant subspace  $E$  the operator  $A$  is bounded, then  $\|A^n x\| \leq C^n \|x\|$  ( $x \in E$ ,  $C = \|A|_E\|$ ); thus, even with  $m_n = C^n$ , inequality (1) will be satisfied by a vector  $x \neq 0$ .

If the operator  $A$  has no spectrum, then its resolvent  $R(\lambda)$  is an entire function, and it is natural to relate the properties of the operator to the growth of  $R(\lambda)$  as  $|\lambda| \rightarrow \infty$ . For the differentiation operator the function  $R(\lambda)$  is of exponential type, bounded on the imaginary axis. We shall take this observation as our starting point.

Let  $A$  be an operator without spectrum. The Taylor series of its resolvent has the form

$$R(\lambda) = \sum_{k=0}^{\infty} A^{-k-1} \lambda^k.$$

Put  $M_A(r) = \max_{|\lambda|=r} \|R(\lambda)\|$  ( $r > 0$ ). The function  $M_A(r)$  as  $r \rightarrow \infty$  grows faster than any power of  $r$ . Introduce the usual growth characteristic of the sequence  $\{m_n\}$ :

$$T(r) = \sup_n \frac{r^{n-1}}{m_n}.$$

**Lemma 1.** *If there exists a vector  $x \neq 0$  for which inequalities (1) hold, then*

$$T(r) \leq M_A(r). \quad (2)$$

**Proof.** We take  $\|x\| = 1$ . Then

$$1 = \|A^{-n}A^n x\| \leq \|A^{-n}\|m_n \leq M_A(r)r^{-n+1}m_n \quad (3)$$

by Cauchy's inequality. Inequality (2) follows from (3).

In view of Lemma 1, from inequalities (1) it will follow that  $x = 0$  whenever the function  $T(r)$ , in some scale, grows faster than  $M_A(r)$ . For example:

**Theorem 1.** If for some  $\rho > 0$

$$\int_1^\infty \frac{\ln T(r)}{r^{1+\rho}} dr = \infty, \quad (4)$$

while

$$\int_1^\infty \frac{\ln M_A(r)}{r^{1+\rho}} dr < \infty, \quad (5)$$

then from inequality (1) it follows that  $x = 0$ .

Condition (4) means that the sequence  $\{m_n\}$  satisfies the classical criterion of quasianalyticity "of order  $\rho$ " (see (6), pp. 55-56). Theorem 1 is sharp in the following sense.

**Theorem 2.** Let  $\psi(t)$  be a positive logarithmically convex function. Suppose that for some  $\rho > 0$

$$\int_1^\infty \frac{\ln \psi(t)}{t^{1+\rho}} dt = \infty. \quad (6)$$

Then there exists an operator  $A$  in a Hilbert space, having no spectrum, and such that: 1)  $\|R(\lambda)\| \leq \psi(|\lambda|)$ ; 2) for some  $x \neq 0$  the sequence  $m_n = \|A^n x\|$  ( $n = 0, 1, 2, \dots$ ) satisfies the quasianalyticity condition (4).

The proof is based on the existence of sequences  $\{a_n\}_0^\infty$  and  $\{m_n\}_{-\infty}^\infty$  of positive numbers for which

$$\sum_{n=0}^{\infty} a_n^{-1} t^n \leq \psi(t) \quad (t \geq 0),$$

$$a_k m_n \leq m_{k+n+1} \quad (n = 0, \pm 1, \pm 2, \dots; k = 0, 1, 2, \dots)$$

and  $\{m_n\}_0^\infty$  satisfies condition (4). In the Hilbert space of sequences  $\xi = \{\xi_n\}_{-\infty}^\infty$  with norm

$$\|\xi\| = \left( \sum_{n=-\infty}^{\infty} |\xi_n|^2 m_n^2 \right)^{1/2}$$

the right-shift operator  $A\xi = \{\xi_{n-1}\}_{n=-\infty}^\infty$  possesses all the required properties. As  $x$  one should take the coordinate unit vector  $\{\delta_{n0}\}$ .

We now formulate the main criterion of quasianalyticity for order  $\rho \leq 1$ .

**Theorem 3.** Let the operator  $A$  have no spectrum and

$$\sigma = \overline{\lim}_{r \rightarrow \infty} r^{-\rho} \ln M_A(r) < \infty \quad (0 < \rho \leq 1).$$

Suppose further that on two rays  $\arg \lambda = \theta_1$ ,  $\arg \lambda = \theta_2$ ,  $\theta_2 - \theta_1 \equiv \pi \rho^{-1} \pmod{2\pi}$ , the estimate  $\|R(\lambda)\| \leq \psi(|\lambda|)$  holds, where  $\psi(t)$  ( $t \geq 0$ ) is a positive continuous nondecreasing function for which

$$\int_1^\infty \frac{\ln \psi(t)}{t^{1+\rho}} \ln t \, dt < \infty. \quad (7)$$

Then, in order that  $x = 0$  follow from inequality (1), it is sufficient, and for  $\sigma > 0$  necessary, that the quasianalyticity condition (4) be fulfilled.

According to Theorem 2, the quasianalyticity criterion (4) is lost when the condition

$$\int_1^\infty \frac{\ln \psi(t)}{t^{1+\rho}} \, dt < \infty, \quad (8)$$

weaker than (7). However, we do not know whether condition (7) in Theorem 3 can be replaced by condition (8).

Let us also note that condition (4) in the case  $\sigma = 0$  is not necessary. The proof of Theorem 3 uses the fact (see (7)) that, for  $\rho = 1$ ,  $\arg \lambda = 0$ , the growth of the resolvent on the real axis can be absorbed by multiplication by a function of the form

$$\varphi(\lambda) = \int_0^a e^{i\lambda t} g(t) dt \quad (a > 0).$$

Owing to this, one can form functions of  $A$  of the form

$$\tilde{\omega}(t; A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \varphi(\lambda) R(\lambda) \omega(\lambda) d\lambda,$$

where  $\omega(\lambda)$  ( $-\infty < \lambda < \infty$ ) may have arbitrary power growth.

In proving sufficiency (in the case  $\rho = 1$ ) we take  $\omega(\lambda) = 1$  and establish that, by virtue of (1),

$$\left\| \frac{d^n}{dt^n} \tilde{\omega}(t; A)x \right\| \leq \text{const} \cdot m_n \|x\|.$$

Then Carleman's theorem implies  $\tilde{\omega}(t; A)x = 0$  ( $t < 0$ ). Reversing (9), we obtain the representation

$$R(\lambda)x = \frac{1}{\varphi(\lambda)} \int_0^b e^{i\lambda t} \tilde{\omega}(t; A)x dt,$$

where  $b > 0$ . By the theorem on addition of indicators ([8], p. 208), the function  $R(\lambda)x$  in the upper half-plane has zero exponential type. An analogous conclusion can be obtained for the lower half-plane. After this, owing to condition (7), the matter reduces to Theorem 1.

In proving necessity, the role of  $\omega(\lambda)$  is played by the Fourier transform of a finite function from the corresponding non-quasianalytic class. The required vector  $x \neq 0$ , for which the estimates (1) hold, is obtained in the form  $\tilde{\omega}(t; A)y$  for some  $y$  and some  $t$ .

If the operator  $A$  has spectrum, the problem of quasianalyticity disappears, but the problem of completeness arises.

**Theorem 4.** *Let the resolvent  $R(\lambda)$  be meromorphic. Put*

$$m_A(r) = \frac{1}{2\pi} \int_0^{2\pi} \ln \|R(re^{i\theta})\| d\theta, \quad \sigma = \overline{\lim}_{r \rightarrow \infty} r^{-\rho} m_A(r) \quad (0 < \rho \leq 1).$$

*Suppose that  $\sigma < \infty$  and that, on the two rays  $\arg \lambda = \theta_1$ ,  $\arg \lambda = \theta_2$ , the conditions of Theorem 3 are satisfied. Suppose, further, that the quasianalyticity condition (4) is satisfied. Then the linear span of the root vectors of the operator  $A$  is dense in the set of vectors satisfying inequalities (1).*

This theorem is proved by the same method as Theorem 3. We emphasize that the case  $\rho > 1$  beyond the scope of Theorems 1, 2 remains uninvestigated.

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Received  
19 XII 1968

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*Note: Figure translations are in progress. See original paper for figures.*

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