BOUNDARY PROPERTIES OF \(n\)-DIMENSIONAL QUASICONFORMAL MAPPINGS

MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 517.53:517.947.42

MATHEMATICS

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BOUNDARY PROPERTIES OF n-DIMENSIONAL QUASICONFORMAL MAPPINGS

(Presented by Academician M. A. Lavrentʹ ev on 19 I 1970)

1. Let R^n be n-dimensional Euclidean space; $\widetilde{R}^n = R^n \cup \{\infty\}$ the corresponding Möbius space; $x = (x_1, \dots, x_n)$ a point (or vector) in R^n; $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ the length of a vector. For an arbitrary domain $\Omega \subseteq R^n$: $L_n(\Omega)$ is the class of measurable functions summable over Ω with exponent n; $W_n^1(\Omega)$ is the set of functions $\varphi(x) \in L_n(\Omega)$ having generalized derivatives $\partial\varphi/\partial x_i \in L_n(\Omega)$ $(i = 1, \dots, n)$; $L_1^n(R^n)$ is the set of functions $\varphi(x)$ representable in the form of a Bessel potential (see ([1,2]))

$$\varphi(x) = (G_1 u)(x) \equiv \int_{R^n} G_1(|x-y|)u(y)\,dy, \qquad u \in L_n(R^n), \tag{1}$$

where $G_1(t)$ is the Bessel kernel of order 1. A vector function $f(x) = (f_1(x), \dots, f_n(x))$ belongs to one of the listed classes if and only if each of its components $f_i(x)$ $(i = 1, \dots, n)$ has this property.

Let A be an arbitrary set in R^n. Consider all possible nonnegative functions $u(x) \in L_n(R^n)$ such that for all $x \in A$

$$(G_1 u)(x) \geq 1.$$

The exact lower bound of the quantities $\left(\|u(x)\|_{L_n(R^n)}\right)$, taken over the set of all such functions $u(x)$, is called the $(1, n)$-capacity (or simply the capacity) of the set A.

This definition of $(1, n)$-capacity is a special case of the definition given in ([3]). In the same work, relations between $(1, n)$-capacity and other capacities are also indicated.

2. Let B be an open ball in R^n, and let S be its boundary. For an arbitrary vector function $f : B \to R^n$, set: $E(f)$ is the set of points on S at which the mapping f has angular boundary values; $E'(f)$ is the set of points on S such that for any point $x_0 \in E'(f)$ there exists a path $\gamma_{x_0} \subset B$, tending

to x_0, along which the mapping f has a limit; $E''(f)$ is the set of points $x_0 \in S$ for each of which there exists a point $y_0 \in \overline{R^n}$ such that

$$\lim_{h \to 0} \frac{1}{h^n} \int_{B_h(x_0)} |f(x) - y_0| \, dx = 0, \tag{2}$$

where $B_h(x_0) = \{x \in B : |x - x_0| < h\}$.

Lemma 1. Let $f : B \to R^n$ be a monotone mapping of the class $W_n^1(B)$, and let $\gamma_{x_0} \subset B$ be an arbitrary path tending to a point $x_0 \in S$. If, for some sequence of points $\{x_n\}$ $(n = 1, 2, \ldots)$ with the properties

$$x_n \in \gamma_{x_0}, \qquad \lim_{n \to \infty} x_n = x_0, \qquad \lim_{n \to \infty} \frac{\operatorname{diam} x_n \tilde{x}_{n+1}}{\rho(x_n, S)} < 1,$$

where $\rho(x_n, S)$ is the distance from the point x_n to S, and if the limit $\lim\limits_{n \to \infty} f(x_n) = y_0$ exists, then the mapping has the same limit along the path γ_{x_0}.

The proof is carried out analogously to the proof of Lemma 3 of paper (4).

Lemma 2 (see (4)). Let $f : B \to R^n$ be a monotone mapping of class $W_n^1(B)$. Then $E(f) = E'(f)$.

Proceeding from condition (2) and applying the preceding lemmas successively, it is not difficult to verify the validity of the following assertion.

Lemma 3. Let $f : B \to R^n$ be a monotone mapping of class $W_n^1(B)$. Then $E''(f) \subseteq E(f)$.

The following Lemma 4 is a simple consequence of Theorem 5.7 of paper (3) (see also (2)).

Lemma 4. Let $f : \dot{R}^n \to R^n$ be a vector-function of class $L_1^n(R^n)$, and let $f_1 = f|_B$ be its restriction to B. Then the set $S \setminus E'''(f_1)$ is a set of zero capacity.

The following theorem is a generalization of the corresponding results from (4–6).

Theorem 1. Let $f : B \to R^n$ be a monotone mapping of class $W_n^1(B)$. Then the set $S \setminus E(f)$ is a set of capacity zero.

We give the idea of the proof. By the extension theorem (see, for example, (1), p. 144) there exists a vector-function $f^* : R^n \to R^n$ of class $W_n^1(R^n)$ such that $f^*|_B = f$. Further, by Theorem 5.1 of paper (3) (see also (2)) there is a vector-function $f^{**} : R^n \to R^n$ of class $L_1^n(R^n)$, coinciding almost everywhere in R^n with f^*.

Consider $f_1 = f^{**}|_B$. For almost all $x \in B$ we have $f_1(x) = f(x)$, and, consequently, $E''(f_1) = E''(f)$. By Lemma 4 the set $S \setminus E''(f_1) = S \setminus E''(f)$ is a set of capacity zero. Applying Lemma 3, we verify the theorem.

Remark. This theorem can be extended (in the corresponding terms) also to monotone mappings of class $W_p^1(B)$, where $p > n - 1$ (see (^{7})).

3. Let $A \subseteq \Omega$ be a set, closed relative to the domain $\Omega \subset R^n$, having no interior points, and let $f : (\Omega \setminus A) \to R^n$ be a quasiconformal mapping, distinct from the identical constant one. To each point $x_0 \in A$ we assign the following sets of values (see (^{8})).

The cluster set $C_{\Omega\setminus A}(f, x_0)$. A value $y \in C_{\Omega\setminus A}(f, x_0)$ if there exists a sequence of points $\{x_n\}$ with the properties

$$x_n \in \Omega \setminus A, \qquad \lim_{n\to\infty} x_n = x_0, \qquad \lim_{n\to\infty} f(x_n) = y.$$

The set of repeated values $R_{\Omega\setminus A}(f, x_0)$. A value $y \in R_{\Omega\setminus A}(f, x_0)$ if there exists a sequence of points $\{x_n\}$ with the properties

$$x_n \in \Omega \setminus A, \qquad \lim_{n\to\infty} x_n = x_0, \qquad f(x_n) = y.$$

In paper (^{10}) we gave certain criteria for the removability of special sets.* The results formulated below concern the structure of the sets $C_{\Omega\setminus A}(f, x_0)$, $R_{\Omega\setminus A}(f, x_0)$ in the case when A is a set of essentially singular points.

Theorem 2. Let $A \subseteq \Omega$ be an arbitrary compact set relative to the domain $\Omega \subset R^n$ of capacity zero. Suppose that $f : (\Omega \setminus A) \to R^n$ is a quasiconformal mapping (nonconstant) having an essential singularity at each point $x_0 \in A$. Then the cluster set

$$C_{\Omega\setminus A}(f, x_0) = \dot{R}^n.$$

For two-dimensional quasiconformal mappings this assertion was proved in (^{9}) (see also (^{8})).

Theorem 3. *Let the hypotheses of Theorem 2 be satisfied. Then in any neighborhood of a point* $x_0 \in A$ *the mapping* f *assumes infinitely often every value* $y \in \overline{R}^n$, *with the possible exception of a set of capacity zero; i.e., the set* $\overline{R}^n \setminus R_{\lambda\setminus A}(f, x_0)$ *is a set of capacity zero.*

In the case where x_0 is an isolated essentially singular point, this theorem is due to Yu. G. Reshetnyak (communication at the Donetsk Colloquium on quasiconformal mappings, 1968).

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* Taking the opportunity, we note that Theorem 1 formulated in (^{10}) is meaningful only for $\alpha = n$. For $\alpha > n$ the class of sets of zero α-capacity is empty.

Received
7 I 1970

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.