ON ONE PROBLEM FOR NONLINEAR EQUATIONS OF PLASMA DYNAMICS

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ON ONE PROBLEM FOR NONLINEAR EQUATIONS OF PLASMA DYNAMICS

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In the present article a theorem is formulated on the existence and uniqueness of the solution of a quasilinear system of equations of a special kind, connected with one problem of plasma dynamics ([1]).

1. Let us consider a plasma cylinder situated in a strong external magnetic field inside a metallic shell of radius r_0. The plasma consists of two sorts of particles with masses m_j, charges e_j, and thermal velocities v_{Tj}. We shall assume that the magnetic field H is directed along the z-axis, depends only on the radial coordinate and is constant in time, while the electric field satisfies the equation $\operatorname{rot} E = 0$ and has a potential $u(r, \varphi, t)$ independent of z. We shall also assume the plasma to be so rarefied that pressure and viscosity may be neglected. In this case the motion of the particles is a rotation along certain circles, the centers of the circles moving in the radial plane with drift velocity ([2],[3])

$$v_j = \frac{c}{H^2}[H \cdot \nabla u] + \frac{m_{jcv_{Tj}}^2}{2e_{jH}^3}[H \cdot \nabla |H|] \tag{1}$$

(c is the speed of light in vacuum). Then the distribution densities of the Larmor centers of the particles $n_j(r, \varphi, t)$, which also do not depend on z, satisfy the continuity equations with velocities of the form (1), and the electric field satisfies Poisson' s equation. Usually the relation $|H'(r)/H| \ll |\nabla n_j/n_j|$ is fulfilled, whence it follows that $|n_j(\nabla v_j)| \ll |(v_j \nabla)n_j|$, and then the behavior of the plasma inside the cylinder is described by the quasilinear system of equations (in dimensionless form)

$$\partial n_1/\partial t + (v_1 \nabla)n_1 = 0, \tag{2}$$

$$\partial n_2/\partial t + (v_2 \nabla)n_2 = 0, \tag{3}$$

$$\Delta u = n_1 - n_2 \tag{4}$$

with the additional conditions

$$n_1|_{t=0} = n_{10}(x), \qquad n_2|_{t=0} = n_{20}(x), \qquad n|_\Gamma = 0. \tag{5}$$

A plane of the form $t = \text{const}$ is a characteristic surface of the system (2)–(4), and the problem belongs to the class of problems with initial conditions on a characteristic surface. The problem formulated describes "groove" oscillations of a plasma cylinder.

The problem (2)–(5) is considered below with some generalizations in comparison with the original physical problem. The two-dimensional domain of variation of x may have arbitrary form; the functions v_j, in contrast to the original formula (1), may be operators of a rather general form. The solution of the problem $n_1(x,t), n_2(x,t), u(x,t)$ is sought as a set of curves depending on the parameter $t \geq 0$, which with respect to the spatial variables belong to the function classes of S. L. Sobolev $W_p^l(G)$ and simultaneously to the Hölder classes $C_{l-1,\nu}(G)$, with the usual norms ([4]).

Theorem. Let, in a two-dimensional, bounded, closed, and convex domain G with an infinitely differentiable boundary Γ, one seek a solution of the system of equations (2)–(4) satisfying conditions (5). Let v_1 and v_2 be certain vector operators acting from the space of scalar functions $C_{2,\nu}(G)$, $\nu < (p-2)/p$, into the space of vector functions $C_{1,\nu}(G)$, continuous and satisfying the Lipschitz condition in u

$$\left\| v_j\left(u^{(1)}\right) - v_j\left(u^{(2)}\right) \right\|_{C_{1,\nu}} \leq L_j \left\| u^{(1)} - u^{(2)} \right\|_{C_{2,\nu}}, \tag{6}$$

with normal components vanishing on the boundary Γ of the domain G and independent of the derivatives of the potential with respect to the parameter t.

Then, if $n_{10}(x), n_{20}(x) \in W_p^1(G)$ $(p > 2)$, then in some neighborhood of $t = 0$ $(0 \leq t \leq t_0)$, determined by the input data, problem (2)–(5) has a solution for $n_j(x,t)$ in $W_p^1(G)$, and $u(x,t)$ in $W_p^3(G)$. If, in addition, the derivatives of $n_{j0}(x)$ have a finite essential supremum, then the solution is unique.

The proof of the existence of a solution is carried out by applying J. Schauder's fixed-point theorem to a certain completely continuous operator constructed from the data of problem (2)–(5).

2. Below we formulate several assertions needed for the proof of the theorem.

First of all, we note that there is a completely continuous embedding of the spaces $W_p^l(G)$ $(p > 2)$ into the spaces $C_{l-1,\nu}(G)$, $\nu < (p-2)/p$ ([4]). Next consider the sequence of problems

$$\Delta u = n_1 - n_2, \quad u|_\Gamma = 0, \quad n_j(x,t) \in C_{0,\nu}(G); \tag{}$$

$$\partial x_j/\partial t = v_j[u(x_j,t)], \quad x_j|_{t=0} = y_j, \quad \{y_j\} = G, \tag{}$$

$$x - x_j(y_j,t) = 0 \quad (j = 1,2);$$

$$\partial \bar{n}_j/\partial t + (v_j \nabla)\bar{n}_j = 0, \quad \bar{n}_j|_{t=0} = n_{j0}(x) \in W^1_p(G). \tag{}$$

Problem (α), as is known, is uniquely solvable in $C_{2,\nu}(G)$, and in this case the inequality

$$\|u(x,t)\|_{C_{2,\nu}} \leq A_1 \left(\|n_1(x,t)\|_{C_{0,\nu}} + \|n_2(x,t)\|_{C_{0,\nu}} \right), \tag{7}$$

holds ($0 < \nu < 1$), where the constant A_1 does not depend on the parameter t. If $n_j(x,t) \in W^1_p(G)$, then the problem is also uniquely solvable in $W^3_p(G)$, and for the indicated norms inequalities of type (9) ([5]) hold. Problem (β) determines the transformation of variables $x_j(y,t) \to y_j(x,t)$. If the conditions of the theorem are fulfilled for $v_j(x,t)$, and in particular $(v_j)_n|_\Gamma = 0$, then problem (β) determines, for all $t \geq 0$, a unique, mutually single-valued, in both directions continuously differentiable mapping of the domain G onto itself, with the boundary going into the boundary and the interior into the interior. The derivatives of the solutions of problem (β) satisfy the equations

$$\frac{\partial x_{ji}}{\partial y_k} = \delta_{ik} + \int_0^t \left(\sum_s \frac{\partial v_{ji}}{\partial x_s} \frac{\partial x_{js}}{\partial y_k} \right) d\tau, \tag{8}$$

the Jacobians $D_j(x/y)$ are solutions of the problem

$$\partial D_j/\partial t = (\operatorname{div} v_j)_{x=x_j(y,t)} D_j, \quad D_j|_{t=0} = 1, \tag{9}$$

are continuous in y and strictly positive for any $t \geq 0$. The inverse functions $y(x,t)$ and their derivatives are computed by the formulas

$$x - x(y,t) = 0,$$

$$\frac{\partial y_1}{\partial x_1} = \frac{1}{D}\frac{\partial x_2}{\partial y_2}, \quad \frac{\partial y_1}{\partial x_2} = -\frac{1}{D}\frac{\partial x_1}{\partial y_2},$$

$$\frac{\partial y_2}{\partial x_1} = -\frac{1}{D}\frac{\partial x_2}{\partial y_1}, \quad \frac{\partial y_2}{\partial x_2} = \frac{1}{D}\frac{\partial x_1}{\partial y_1}. \tag{10}$$

In (10) the indices indicate the number of the coordinate of the point x; the index j, which relates formula (10) to equations (2) or (3), has been omitted. Formulas (8)–(10) make it possible to obtain all the estimates needed for solutions of problem (β).

Problem (γ) is also uniquely solvable in $W_p^1(G)$, and the solution can be written, using problem (β), in the form

$$\bar{n}_j(x,t) \equiv n_{j0}(y_j(x,t)). \tag{11}$$

3. We construct the operators B_j in the following way:

$$\bar{n}_j(x,t) = B_j n_j(x,t) \qquad (j = 1, 2). \tag{12}$$

Let $n_j(x,t)$ be arbitrary curves, equal to $n_{j0}(x)$ for $t = 0$, lying in some bounded set of the space $C_{0,\nu}(G)$ and continuous in t. Substitute them into the right-hand side of problem (α) and obtain the solution $u(x,t) \in C_{2,\nu}(G)$. Next, from the solution u thus found, using the prescribed operators $v_j[u(x,t)]$, solve problem (β), and, finally, by means of this solution, determine from problem (γ) the functions $\bar{n}_j(x,t)$ standing on the left-hand side of (12). Relying on the constructions of item 2, it is not difficult to verify that the operators B_j act from the complete space of curves continuous in t, lying in $C_{0,\nu}(G)$, into the space of curves lying in $W_p^1(G)$, compact with respect to $C_{0,\nu}(G)$. Using the equations and the additional conditions of problems (α), (β), (γ), as well as formulas (6)–(11), one can prove that the curves $\bar{n}_j(x,t)$ are equicontinuous in t in $W_p^1(G)$ for any bounded set of curves $n_j(x,t)$ lying in $C_{0,\nu}(G)$, and therefore the operators B_j are compact ([6]). It is not difficult to prove their continuity as well.

Consider the system of numerical inequalities for the quantities ψ and χ (obtained with the aid of a priori estimates for solutions of problems (2)–(5))

$$\psi \le A_2 \max_{j=1,2} \|n_{j0}(x)\|_{W_p^1} e^{\sigma t \chi},$$

$$\chi \le A_1 \psi. \tag{13}$$

Here A_1, A_2, σ are constants depending only on the geometric characteristics of the problem and the summability exponent of the functions p. It is not difficult to verify that system (13) determines, for some interval of variation of t ($0 \le t \le t_0$), a domain of values of ψ, χ lying entirely in the rectangle $\{0 \le \psi \le A_3,\ 0 \le \chi \le A_4\}$. Let the domain of definition for the operators

B_j be the set of curves lying in the closed ball $\|n_j(x,t)\|_{C_{0,\nu}} \leq A_3$. Then, using the formulas of item 2, with the aid of (13) one can prove that the completely continuous operators B_j map the set under consideration into its own subset and, consequently, by Schauder' s theorem have a fixed point. Since the operators B_j take any curve continuous in t from $C_{0,\nu}(G)$ into a curve from $W_p^1(G)$, all fixed points are curves from $W_p^1(G)$, and by the construction of the operators B_j they are solutions of problem (2)–(5).

4. We pass to the proof of uniqueness of the solution. Let

$$M = \max_{j=1,2}\left\{\operatorname*{vrai\,sup}_{x\in G}|\nabla n_{j0}(x)|\right\}$$

and let there exist two solutions of the problem $n_j(x,t)$, $\overset{r}{y}_j(x,t)$, $\overset{r}{u}(x,t)$ $(r=1,2)$. From consideration of problem (γ) it follows that any function $\overset{r}{n}_j(x,t)$ can be represented in the form (11). Then from (13), for any r, it follows that

$$\max_{r,\ j=1,2}\left\{\operatorname*{vrai\,sup}_{(x,t)}\left|\nabla n_{j0}(\overset{r}{y}_j(x,t))\right|\right\} \leq MA_5 \leq M_1. \tag{14}$$

Since the derivative (in the sense of S. L. Sobolev) is unique, and differentiability of the functions under consideration is guaranteed by the theorem of exis–

existence, it is sufficient to prove that the maximum of the modulus of the difference of the solutions is equal to zero for all t in the domain of existence of the solution. Using the results of item 2, it is not hard to verify that the inequalities

$$\max_{x\in G}\left|n_{j0}\left({}^1y_j(x,t)\right) - n_{j0}\left({}^2y_j(x,t)\right)\right| \leq M_1\left|{}^1y_j(x,t) - {}^2y_j(x,t)\right| \leq$$

$$\leq A_6M_1\max_{(x,t,j,r)}\left|\nabla y_j^r\right|\int_0^t \max\left|{}^1v_j(x,\tau) - {}^2v_j(x,\tau)\right|\,d\tau \leq$$

$$\leq A_7\int_0^t \max\left|n_{j0}\left({}^1y_j(x,\tau)\right) - n_{j0}\left({}^2y_j(x,\tau)\right)\right|\,d\tau,$$

hold; from these, for some $t_1 \leq t_0$ $(0 \leq t \leq t_1)$, the uniqueness of the solution of problem (2)–(5) follows. From (13) and (14) follows the possibility of extending the uniqueness theorem to the whole domain of existence of the solution.

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Note: Figure translations are in progress. See original paper for figures.