ON THE STRUCTURE OF CONTROL LAWS ENSURING ASYMPTOTIC STABILITY OF CONTROL SYSTEMS WITH AN UNSTABLE PLANT

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Abstract

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ON THE STRUCTURE OF CONTROL LAWS ENSURING ASYMPTOTIC STABILITY OF CONTROL SYSTEMS WITH AN UNSTABLE PLANT

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Let a control system be described by the equation

$$\dot{x} = f(x, u), \tag{1}$$

where x is an n-dimensional vector of the phase coordinates of the system; $u = u(x)$ is an m-dimensional vector-valued function of the system coordinates (the control law) with values in a set U, belonging to a class D of admissible control laws such that, if some control law $u_1(x) \in D$ is defined in an open subset Ω_1 of the phase space, and $u_2(x) \in D$ in a subset Ω_2 (also open), then the control law $u_0(x)$, equal to $u_1(x)$ for $x \in \Omega_1$ and to $u_2(x)$ for $x \in \Omega_2 \setminus \Omega_1$, is also admissible: $u_0(x) \in D$. Here $\Omega_2 \setminus \Omega_1$ denotes the complement of the set Ω_1 in Ω_2. We shall assume that the n-dimensional vector-valued function $f(x, u)$ is such that system (1) satisfies the conditions of existence and uniqueness of the solution for $u(x) \in D$.

Denote by $\Omega[u(x)]$ the domain of attraction (the domain of asymptotic stability) of the origin of the coordinates of system (1) under the control law $u(x)$. If the origin is unstable, set $\Omega = \emptyset$.

Definition 1. A control law $\hat{u}(x)$ will be called **optimal with respect to stability** if, for any admissible control law $u(x)$, the condition $\Omega[u(x)] \subset \Omega[\hat{u}(x)]$ is satisfied.

Theorem 1. *Among the admissible control laws there exists one that is optimal with respect to stability.*

Proof. We shall prove that there exists a control law $\hat{u}(x)$ such that

$$\Omega[\hat{u}(x)] = \bigcup_{u \in D} \Omega[u(x)].$$

The union $\Omega^* = \bigcup_{u\in D} \Omega[u(x)]$ is open, since each $\Omega[u(x)]$ is open. The phase space X of system (1) has a countable base, and the system of sets $\{\Omega[u] : u \in D\}$ is an open covering of $\Omega^* \subset X$, and therefore contains an at most countable subcovering $\Omega_1 = \Omega[u_1(x)]$, $\Omega_2 = \Omega[u_2(x)]$, ... The control law

$$\hat{u}(x) = u_i(x) \quad \text{for } x \in \Omega_i \setminus \bigcup_{j=1}^{i-1} \Omega_j, \quad i = 1, 2, \dots,$$

belongs to the class D, is defined on Ω^*, and is optimal in the sense of Definition 1.

Obviously, as the class of admissible control laws one may take the class of piecewise-continuous, piecewise-constant, or piecewise-linear functions in each coordinate.

Suppose there exists a nondegenerate linear transformation with matrix J of the system coordinates, bringing system (1) to the form

$$\dot{y}^+ = f^+(y^+, u), \qquad (A)$$

$$\dot{y}^- = f^-(y^+, y^-, u), \qquad (B) \tag{2}$$

where $y = (y^+, y^-) = Jx$, $x = J^{-1}y$, and f^+, f^- are, respectively, k- and $(n-k)$-dimensional vector-valued functions.

In the case where the control law $u(y)$ depends only on the vector y^+, $u(y) = u(y^+)$, equation (2A) may be regarded as the equation of a certain dynamical system of order k, whose phase space we denote by Y^+. On the other hand, equation (2B), for $y^+ = 0$ and $u(y) = u(y^-)$, may be regarded as the equation of a dynamical system of order $(n-k)$, whose phase space we denote by Y^-. Obviously, the phase space Y of system (2) is equal to the topological product of the spaces Y^+ and Y^-: $Y = Y^+ \times Y^-$. We shall call these dynamical systems subsystems of system (2).

Under the adopted conventions, the following theorem holds, which makes it possible to reduce the problem of synthesizing a stability-optimal control of system (2) to the synthesis of a control law for a system of lower order.

Theorem 2. *Suppose that, for system (2), the following conditions are satisfied:*

1. There exists a stability-optimal control law $\hat{u}_+$ for subsystem (A), which brings any point $y^+ \in \Omega^+[\hat{u}(y^+)] \subseteq Y^+$ to the origin of the phase space Y^+ in finite time.

2. Subsystem (2B) of system (2) is asymptotically stable for $u = 0$ and $y^+ = 0$, so that for any initial conditions $y^- \in Y^-$ one always has

$$\lim_{t\to\infty} y^-(t) = 0.$$

Then the domain of asymptotic stability $\Omega[\hat{u}(y)]$ corresponding to the stability-optimal control of system (2) is equal to

$$\Omega[\hat{u}(y)] = \Omega^+[\hat{u}_+(y^+)] \times Y^-.$$

Proof. By assumption, if $y \in \Omega[\hat{u}]$, then

$$\lim_{t\to\infty} y(t) = 0;$$

therefore,

$$\lim_{t\to\infty} y^+(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} y^-(t) = 0,$$

where the vectors $y^+ = \mathrm{Pr}_{Y^+}\, y$, $y^- = \mathrm{Pr}_{Y^-}\, y$ are the projections of the vector y onto the phase spaces Y^+ and Y^- of subsystems (2A) and (2B), respectively; hence, if $y \in \Omega[\hat{u}]$, then $y^+ \in \Omega^+[\hat{u}_+]$, and consequently,

$$\Omega[\hat{u}(y)] \subset \Omega^+[\hat{u}_+(y^+)] \times Y^-.$$

To prove the theorem it remains to show that there exists a control law $u_0(y)$ such that

$$\Omega[u_0(y)] = \Omega^+[\hat{u}_+(y^+)] \times Y^-.$$

If $u_+(y^+)$ is a stability-optimal control law that brings any point $y^+ \in \Omega^+[\hat{u}_+(y^+)]$ to the origin in finite time, then set

$$u_0(y) = \begin{cases} \hat{u}_+(y^+) & \text{for } y^+ \neq 0, \\ 0 & \text{for } y^+ = 0. \end{cases}$$

In view of the properties of system (2B),

$$\lim_{t\to\infty} y^-(t) = 0$$

for $u = u_0(y)$ and for any initial phase states y for which $y^+ \in \Omega[\hat{u}_+]$.

If subsystem (2A) is called the unstable part of system (2), then the result obtained means that, in order to ensure the stability of system (2), it is sufficient to control its unstable part in an optimal manner. In this case the optimal control law for system (1) is equal to

$$\hat{u}(x) = u_0(Jx).$$

Example 1. Let system (1) be a linear system of the form

$$\dot{x} = Ax + Bu,$$

where A and B are constant matrices, and U is a convex closed polyhedron; moreover, the general-position condition (1) is satisfied, and the roots of the characteristic polynomial of the matrix A are real and distinct.

As J we choose the transformation matrix that brings A to Jordan canonical form. Then equation (2A) takes the form

$$\dot{y}^+ = A^+ y^+ + B^+ u,$$

where A^+ is a diagonal matrix with nonnegative elements.

As the stability-optimal control law $\hat{u}_+(y^+)$ one may take the control law corresponding to the time-optimal control, which always exists whenever there exists at least one control bringing the point y^+ to the origin ([1]).

In this case, to the phase space Y^- there will correspond a certain linear manifold X^- in the phase space of system (1), such that for any initial states $x \in X^-$ and with $u(x) = 0$ one always has

$$\lim_{t\to\infty} x(t) = 0.$$

Example 2. The unstable nonlinear second-order system

$$\dot{x}_1 = x_1 + x_2 - f(a_1 x_1 + a_2 x_2) + b_1 u,$$

$$\dot{x}_2 = f(a_1 x_1 + a_2 x_2) + b_2 u,$$

where $a_1 \neq a_2$, and $f(z)$ is a nonlinear function such that $zf(z) > 0$ for $z \neq 0$, is reduced by the linear transformation $y_1 = x_1 + x_2$ and $y_2 = a_1 x_1 + a_2 x_2$ to the form

$$\dot{y}_1 = y_1 + (b_1 + b_2)u,$$

$$\dot{y}_2 = a_1 y_1 + (a_2 - a_1) f(y_2) + (a_1 b_1 + a_2 b_2)u.$$

For $a_1 > a_2$ the nonlinear first-order system $\dot{y}_2 = (a_2 - a_1) f(y_2)$ is asymptotically stable, so that the transformed system satisfies the conditions of Theorem 2. If the control is bounded in modulus, $|u| \leq c$, then as a stability-optimal control one may take

$$u = -c \operatorname{sgn}\big[(b_1 + b_2) y_1\big],$$

or, equivalently,

$$u = -c \operatorname{sgn}\big[(b_1 + b_2)(x_1 + x_2)\big].$$

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CITED LITERATURE

1. V. G. Boltyanskii, *Mathematical Methods of Optimal Control*, Nauka, 1966.

Note: Figure translations are in progress. See original paper for figures.