EXISTENCE AND REGULARITY OF THE LIMITING GIBBS STATE FOR ONE-DIMENSIONAL CONTINUOUS SYSTEMS OF QUANTUM STATISTICAL MECHANICS

MATHEMATICS

1970

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Abstract

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UDC 517.433

MATHEMATICS

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EXISTENCE AND REGULARITY OF THE LIMITING GIBBS STATE FOR ONE-DIMENSIONAL CONTINUOUS SYSTEMS OF QUANTUM STATISTICAL MECHANICS

(Presented by Academician A. N. Kolmogorov on 13 V 1970)

1. In a recent work of H. Araki ([1]) the limiting Gibbs state was constructed for one-dimensional spin systems. In the present note the limiting Gibbs state is constructed for one-dimensional continuous quantum systems. We assume that the interaction is binary, with a finite potential $V(r)$ possessing a hard core of radius $d > 0$.

2. Let $\Omega = (a_1, a_2)$ be a finite interval on the line R^1. Introduce the Fock spaces $\mathcal{H}_B(\Omega)$ of bosons and $\mathcal{H}_F(\Omega)$ of fermions and the particle-number operator N (see, for example, ([2]) *). The Hamiltonian H is a self-adjoint extension of the symmetric operator $\hat{H}$ generated by the Schrödinger equation:

$$\hat{H} f(x_1, \dots, x_n) = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f(x_1, \dots, x_n) + \sum_{j \neq k} V(|x_j - x_k|) f(x_1, \dots, x_n),$$

$$x_j \in \Omega, \quad j = 1, \dots, n, \quad \min_{j \neq k} |x_j - x_k| > d, \quad n = 1, 2, \dots$$

We consider self-adjoint extensions H of the operator $\hat{H}$ determined by the following types of boundary conditions on the boundary $\partial\Omega^n$ of the cube $\Omega^n = \Omega \times \dots \times \Omega$:

$$f|_{\partial\Omega^n} = 0 \quad \text{and} \quad \frac{\partial}{\partial\nu} f|_{\partial\Omega^n} = 0,$$

where ν is the outward normal.

The hard-core condition for $V(r)$ means that both extensions are considered in the subspaces $\mathcal{H}_B^{(d)}(\Omega) \subset \mathcal{H}_B(\Omega)$ and $\mathcal{H}_F^{(d)}(\Omega) \subset \mathcal{H}_F(\Omega)$, consisting of chains of functions

$$f = \{f_0, f_1(x), f_2(x_1, x_2), ...\},$$

vanishing when $\min_{j \neq k} |x_j - x_k| \leq d$. The symbol (d) will be omitted below. The limiting objects described in Theorems 1, 3, and 4 do not depend on the choice of extension.

The grand canonical Gibbs ensemble in Ω generates a linear positive normalized functional (state) $G_{\zeta,\beta,\Omega}$ on the C^*-algebras $\mathfrak{A}_B(\Omega)$ and $\mathfrak{A}_F(\Omega)$ of bounded operators in $\mathcal{H}_B(\Omega)$ and $\mathcal{H}_F(\Omega)$, respectively: for $A \in \mathfrak{A}_B(\Omega)$ $(\mathfrak{A}_F(\Omega))$

$$G_{\zeta,\beta,\Omega}(A) = \operatorname{tr} A\rho_{\zeta,\beta,\Omega},$$

where

$$\rho_{\zeta,\beta,\Omega} = \left[\operatorname{tr} \zeta^N \exp(-\beta H)\right]^{-1} \zeta_N \exp(-\beta H)$$

is a positive definite nuclear operator in $\mathcal{H}_B(\Omega)$ (respectively in $\mathcal{H}_F(\Omega)$) with trace 1 (the density matrix). Under our assumptions on $V(r)$, this functional exists for all $\zeta > 0$ and $\beta > 0$.

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* The spaces $\mathcal{H}_B(\Omega)$ and $\mathcal{H}_F(\Omega)$ can be introduced for any Lebesgue-measurable subset of the line.

3. Let $\Omega \to \infty$ be the set of finite intervals directed by inclusion. Then the c^*-algebras $\mathfrak{A}_B(\Omega)$ and $\mathfrak{A}_F(\Omega)$ form direct spectra of c^*-algebras (see ([4])). Let $\mathfrak{A}_B^{(0)}$ and $\mathfrak{A}_F^{(0)}$ be the corresponding inductive limits, and let

$$\mathfrak{A}_B = \overline{\mathfrak{A}}_B^{(0)}, \qquad \mathfrak{A}_F = \overline{\mathfrak{A}}_F^{(0)}$$

(the bar denotes closure in the norm).

The state $G_{\zeta,\beta,\Omega}$, for fixed Ω, may be regarded as given on families of c^*-algebras $\{\mathfrak{A}_B(\Omega')\}$ and $\{\mathfrak{A}_F(\Omega')\}$, where $\Omega' \subset \Omega$.*

Theorem 1. *For all $\zeta > 0$ and $\beta > 0$, for any $A \in \mathfrak{A}_B^{(0)}$ $(\mathfrak{A}_F^{(0)})$, there exists the limit*

$$\lim_{\Omega \to \infty} G_{\zeta,\beta,\Omega}(A) = G_{\zeta,\beta}^{(0)}(A).$$

From Theorem 1 and theorems on the extension of positive linear functionals it follows that on the c^*-algebra $\mathfrak{A}_B$ $(\mathfrak{A}_F)$ there exists a unique state $G_{\zeta,\beta}$ coinciding on $\mathfrak{A}_B^{(0)}$ $(\mathfrak{A}_F^{(0)})$ with $G_{\zeta,\beta}^{(0)}$. The state $G_{\zeta,\beta}$ is called the limiting Gibbs state.

4. In the c^*-algebras $\mathfrak{A}_B$ and $\mathfrak{A}_F$ there acts a one-parameter group $\{T_\tau, -\infty < \tau < +\infty\}$ of spatial shifts (translations) ([1]). The limiting state $G_{\zeta,\beta}$ is invariant under the action of this group and has the regularity property described in the following theorem:

Theorem 2. *Let* $A_1 \in \mathfrak{A}_B(\Omega_1)$, $A_2 \in \mathfrak{A}_B(\Omega_2)$ $(A_1 \in \mathfrak{A}_F(\Omega_1),\ A_2 \in \mathfrak{A}_F(\Omega_2))$. *Then*

$$\left|G_{\zeta,\beta}(A_1 \cdot (T_\tau A_2)) - G_{\zeta,\beta}(A_1) G_{\zeta,\beta}(A_2)\right| \leq \|A_1\| \cdot \|A_2\| \gamma(\tau, \Omega_1, \Omega_2), \qquad (1)$$

where (for any fixed $\zeta > 0$ *and* $\beta > 0$*)*

$$\lim_{|\tau| \to \infty} \gamma(\tau, \Omega_1, \Omega_2) = 0.$$

5. The theorem given below establishes the impossibility of phase transitions (in the sense of Ehrenfest ([6])) with respect to ζ in the systems under consideration.

Theorem 3. *The limit*

$$\lim_{\Omega \to \infty} \frac{1}{a_2 - a_1} \ln \operatorname{tr} \zeta^N \exp(-\beta H) = \varphi(\zeta, \beta)$$

defines, for all $\beta > 0$*, an infinitely differentiable function of the variable* $\zeta > 0$.

Similar results are contained in ([5]).

6. **Scheme of the proof of Theorems 1 and 2.** Let $\Omega_0 \subset \Omega$. The reduced density matrix $\rho^{(\Omega_0)}_{\zeta,\beta,\Omega}$ is defined by the condition

$$G_{\zeta,\beta,\Omega}\left(\pi_\Omega^{\Omega_0} A\right) = \operatorname{tr} A \rho^{(\Omega_0)}_{\zeta,\beta,\Omega} \qquad (2)$$

and is a positive-definite nuclear operator in $\mathcal{H}_B(\Omega_0)$ $(\mathcal{H}_F(\Omega_0))$ with trace 1.** We denote the kernel of this operator by $\rho^{(\Omega_0)}_{\zeta,\beta,\Omega}(\bar{x}, \bar{y})$. Here

$$\bar{x} = (x_1, \dots, x_n), \qquad \bar{y} = (y_1, \dots, y_n); \qquad x_j,\ y_j \in \Omega_0, \qquad j = 1, \dots, n;$$

$$\min[|x_j - x_k|,\ |y_j - y_k|] > d; \qquad n = 1, 2, \dots$$

* In this case, in both instances, if $\{\pi_\Omega^{\Omega'}\}$ are homomorphisms of the direct spectrum $(\Omega' \subset \Omega)$, then for $A \in \mathfrak{A}(\Omega')$

$$G_{\zeta,\beta,\Omega}\left(\pi_\Omega^{\Omega'} A\right) = G_{\zeta,\beta,\Omega}(A \otimes E),$$

where E is the identity operator in $\mathcal{H}(\Omega \setminus \Omega')$ (see ([4])).

** The operator $\rho^{(\Omega_0)}_{\zeta,\beta,\Omega}$ is determined uniquely by condition (2). It can be shown that it has the form

$$\rho^{(\Omega_0)}_{\zeta,\beta,\Omega} = \left[\operatorname{tr} \zeta^N \exp(-\beta H)\right]^{-1} \operatorname{tr}_{\mathcal{H}(\Omega \setminus \Omega_0)} \zeta^N \exp(-\beta H).$$

Theorem 4. *For all $\zeta > 0$ and $\beta > 0$, for any Ω_0 there exists the limit*

$$\lim_{\Omega\to\infty} \rho^{(\Omega_0)}_{\zeta,\beta,\Omega}(\bar{x},\bar{y}) = \rho^{(\Omega_0)}_{\zeta,\beta}(\bar{x},\bar{y}),$$

uniformly for fixed $[l_1, l_2]$ in $\Omega_0 \subset [l_1, l_2]$ and $\bar{x}, \bar{y} \in \Omega_0^n$, $n = 1, 2, \ldots$.

The limiting kernel $\rho^{(\Omega_0)}_{\zeta,\beta}(\bar{x},\bar{y})$ defines in $\mathcal{H}_B(\Omega_0)$, $\mathcal{H}_F(\Omega_0)$ a positive definite kernel operator $\rho^{(\Omega_0)}_{\zeta,\beta}$ with trace 1. The operators $\rho^{(\Omega_0)}_{\zeta,\beta}$ and $\rho^{(\Omega_0')}_{\zeta,\beta}$, for $\Omega_0 \subset \Omega_0'$, are related by

$$\rho^{(\Omega_0)}_{\zeta,\beta} = \operatorname{tr}_{\mathcal{H}(\Omega_0'\setminus\Omega_0)} \rho^{(\Omega_0')}_{\zeta,\beta}.$$

The method of proof of Theorem 4 is a modification of the well-known matrix method (transfer-matrix method) in classical statistical mechanics.

Theorem 1 is a consequence of Theorem 4 and Lemma 1.

Lemma 1. *Let $\rho_n(x, y)$ be a sequence of kernels defining positive definite kernel operators ρ_n with trace 1 in the Hilbert space $L^2(M, \mu)$, $\mu(M) < \infty$. Suppose that, uniformly in $x, y \in M$, there exists the limit*

$$\lim_{n\to\infty} \rho_n(x, y) = \rho(x, y),$$

which defines a positive definite kernel operator ρ with trace 1. Then

$$\lim_{n\to\infty} \|\rho_n - \rho\|_1 = 0,$$

where $\|A\|_1 = \operatorname{tr}\sqrt{AA^*}$.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ be the sequence of eigenvalues of the operator ρ, and $e_i(x)$, $i = 1, 2, \ldots$, the corresponding eigenvectors. From (3) it follows that

$$\lim_{n\to\infty} \sum_{i,j} \big((\rho_n e_i, e_j) - \lambda_i \delta_{ij}\big)^2 = 0.$$

We shall now show that the sequence $\{\rho_n\}$ is compact in the Banach space $\mathfrak{P}$ of kernel operators in $L^2(M, \mu)$ with norm $\|\cdot\|_1$. To this end we use the compactness criterion in Banach spaces with basis (3). As a basis in $\mathfrak{P}$ it is natural to take the system $\{E_{ij}\}$ of matrix units for the basis $\{e_i(x),\ i = 1, 2, \ldots\}$. Let

$$\rho^{(i_0)}_n = \sum_{i,j<i_0} (\rho_n e_i, e_j) E_{ij}, \qquad \bar{\rho}^{(i_0)}_n = \sum_{i,j\geq i_0} (\rho_n e_i, e_j) E_{ij},$$

$$\tilde{\rho}^{(i_0)}_n = \sum_{i=1}^{i_0-1} \sum_{j=i_0}^{\infty} (\rho_n e_i, e_j) E_{ij}.$$

Obviously, $\rho^{(i_0)}_n$ and $\bar{\rho}^{(i_0)}_n$ are positive definite operators,

$$\|\rho^{(i_0)}_n\|_1 + \|\bar{\rho}^{(i_0)}_n\|_1 = 1,$$

and also

$$\rho_n = \rho_n^{(i_0)} + \bar{\rho}_n^{(i_0)} + \tilde{\rho}_n^{(i_0)*} + \tilde{\rho}_n^{(i_0)}.$$

Take an arbitrary $\varepsilon > 0$ and choose $i_0 = i_0(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ so that

$$\sum_{i \geq i_0} \lambda_i < \varepsilon/4,$$

and, for $n > n_0(\varepsilon)$,

$$\left| \|\rho_n^{(i_0)}\|_1 - \sum_{i=1}^{i_0-1} \lambda_i \right| < \frac{\varepsilon}{4}, \qquad \sum_{i \neq j} \left((\rho_n e_i, e_j)\right)^2 < \frac{\varepsilon^2}{16 i_0^2}.$$

* A similar assertion was obtained by I. D. Novikov.

Then, for $n > n_0$,

$$\left\|\rho_n - \rho_n^{(i_0)}\right\|_1 \leq \left\|\rho_n^{(i_0)}\right\|_1 + 2\left\|\tilde{\rho}_n^{(i_0)}\right\|_1 = 1 - \left\|\rho_n^{(i_0)}\right\|_1 + 2\left\|\tilde{\rho}_n^{(i_0)}\right\|_1 \leq$$

$$\leq 1 - \sum_{i=1}^{i_0-1} \lambda_i + \frac{\varepsilon}{4} + 2\left\|\tilde{\rho}_n^{(i_0)}\right\|_1 < \frac{\varepsilon}{2} + 2\left\|\tilde{\rho}_n^{(i_0)}\right\|_1.$$

The proof of the lemma is completed by the obvious observation that

$$\left\|\tilde{\rho}_n^{(i_0)}\right\|_1 \leq \sum_{i=1}^{i_0-1} \left[\sum_{j=i_0}^{\infty} \left((\rho_n e_i, e_j)\right)^2\right]^{1/2} < i_0 \left[\sum_{i \neq j} \left((\rho_n e_i, e_j)\right)^2\right]^{1/2}.$$

Theorem 2 is a consequence of Lemma 1 and of Theorem 5 stated below, which establishes the so-called cluster property of the limiting kernels $\rho_{\xi,\beta}^{(\Omega_0,\Omega_0')}(\bar{x},\bar{y})$. Let $\bar{x},\bar{y} \in \Omega^l$; $\bar{x}',\bar{y}' \in {\Omega_0'}^{l'}$; $T_\tau \bar{x}' = (x_1'+\tau,\dots,x_l'+\tau)$; $T_\tau \bar{y}' = (y_1'+\tau,\dots,y_l'+\tau)$.

Theorem 5. *The inequality holds*

$$\left|\rho_{\xi,\beta}^{(\Omega_0 \cup T_\tau \Omega_0')}(\bar{x} \cup T_\tau \bar{x}', \bar{y} \cup T_\tau \bar{y}') - \rho_{\xi,\beta}^{(\Omega_0)}(\bar{x},\bar{y})\,\rho_{\xi,\beta}^{(\Omega_0')}(\bar{x}',\bar{y}')\right| \leq \bar{\gamma}(\tau,\Omega_0,\Omega_0'),$$

where

$$\lim_{|\tau|\to\infty} |\tau|^\alpha \bar{\gamma}(\tau,\Omega_0,\Omega_0') = 0 \quad \text{for all } \alpha > 0.$$

The proof of Theorem 3 rests on Theorems 4 and 5 and is carried out by the standard method of the so-called correlation functions.

The author thanks Ya. G. Sinai for useful discussions of the results and comments during the preparation of the note.

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Received
29 IV 1970

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Note: Figure translations are in progress. See original paper for figures.