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Abstract

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MATHEMATICS

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ESTIMATES OF THE SOLUTION OF A BOUNDARY-VALUE PROBLEM WITH A COMPLEX PARAMETER FOR AN ELLIPTIC SYSTEM OF SECOND ORDER

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In connection with the application of the contour-integral method ⁽¹⁾ to the solution of mixed problems for parabolic systems containing in the boundary condition a derivative with respect to time, the present note is devoted to the solution of a boundary-value problem with complex parameter λ for an elliptic system of second order and to obtaining estimates for this solution.

In a certain three-dimensional domain D with boundary Γ , the problem is considered of finding a solution of the equation

$$A(x)\Delta u + \sum_{i=1}^3 A_i(x) \frac{\partial u}{\partial x_i} + (A_0(x) - \lambda^2) u = \Phi(x) \quad (1)$$

under the boundary condition

$$\lim_{x \rightarrow z} \left\{ (\alpha_0(z) + \lambda^2 \alpha_1(z)) \frac{d}{dn_z} + (\beta_0(z) + \lambda^2 \alpha_1(z) \beta_1(z)) \right\} u(x, \lambda) = \psi(z, \lambda), \quad z \in \Gamma, \quad (2)$$

where $A(x), A_i(x)$ ($i = 0, 1, 2, 3$) are square matrices of order m ; $\Phi(x)$ is a vector-function defined in D ; $\alpha_i(z), \beta_i(z)$ ($i = 0, 1$) are square matrices of order m , continuous on Γ ; $\psi(z, \lambda)$ is a vector-function defined on Γ ; Δ is the Laplace operator with respect to the point $x = (x_1, x_2, x_3)$.

It is assumed that the following conditions are satisfied.

1°. The matrices $A(x), A_i(x)$ ($i = 0, 1, 2, 3$) have bounded continuous derivatives in the domain D , and the roots $\nu_i(x)$ of the characteristic equation

$$\det(A(x) + \nu E) = 0$$

for all x in the closed domain D have constant multiplicities m_i and strictly negative real parts*.

2°. The matrices $\alpha_i(z), \beta_i(z)$ ($i = 0, 1$) are continuous on Γ , and for sufficiently large complex λ the matrix

$$[\alpha_0(z) + \lambda^2 \alpha_1(z)]^{-1} [\beta_0(z) + \lambda^2 \alpha_1(z) \beta_1(z)]$$

is bounded for $z \in \Gamma$ by a number independent of λ . The vector-function $\psi(z, \lambda)$, analytic in λ in the domain R_δ , tending to zero as $|\lambda| \rightarrow +\infty$, and continuous in z on Γ , where R_δ is the domain of values λ satisfying the inequalities

$$|\lambda| \geq R, \quad |\arg \lambda| \leq \pi/4 + \delta \quad (R_\delta)$$

for sufficiently large R .

In note (2), under condition 1°, the existence was proved of a fundamental matrix $P(x, \xi, \lambda)$, analytic in λ in the domain R_δ , for which it is obtai-

* From this condition there follows the parabolicity of the corresponding dynamical system.

appropriate estimates were also obtained. From the estimates of the fundamental matrix obtained in (2), it follows that for the derivative with respect to the interior normal to Γ at the point $z \in \Gamma$ of the simple-layer potential

$$u(x, \lambda) = \int_{\Gamma} P(x, y, \lambda) \mu(y) d\Gamma_y \quad (3)$$

the jump formulas hold

$$\left(\frac{du(z, \lambda)}{dn_z} \right)_i = -\frac{1}{2} A^{-1}(z) \mu(z) + \int_{\Gamma} \frac{dP(z, y, \lambda)}{dn_z} \mu(y) d\Gamma_y, \quad (4)$$

$$\left(\frac{du(z, \lambda)}{dn_z} \right)_e = \frac{1}{2} A^{-1}(z) \mu(z) + \int_{\Gamma} \frac{dP(z, y, \lambda)}{dn_z} \mu(y) d\Gamma_y. \quad (5)$$

Analogous jump formulas hold for the double-layer potential.

By virtue of formulas (4), (5), the determination of the solution of problem (1) –(2) for the corresponding homogeneous system in the form of the simple-layer potential (3) leads to the integral equation

$$\mu(z, \lambda) = \tilde{\psi}(z, \lambda) + \int_{\Gamma} K(z, y, \lambda) \mu(y, \lambda) d\Gamma_y, \quad (6)$$

where

$$\tilde{\psi}(z, \lambda) = -2A(z)(A_0(z) + \lambda^2 a_1(z))^{-1} \psi(z, \lambda), \quad (7)$$

$$K(z, y, \lambda) = 2A(z) \left[\frac{d}{dn_z} + (\alpha_0(z) + \lambda^2 \alpha_1(z))^{-1} (\beta_0(z) + \lambda^2 \alpha_1(z) \beta_1(z)) \right] P(z, y, \lambda).$$

According to the theorem of the note (2) on the fundamental matrix $P(z, y, \lambda)$, under conditions 1°–3°, for the kernel $K(z, y, \lambda)$ in the domain R_δ the estimate

$$|K(z, y, \lambda)| \leq CB \exp[-\varepsilon|\lambda||z - y|/|z - y|^{2-\alpha}]; \quad (8)$$

is valid. Here C, ε are certain positive constants; B is a square matrix of order m composed of ones; ε is a certain positive constant; α is a Lyapunov number; $|z - y|$ is the length of the vector $z - y$; inequality (8) is understood in the sense that it holds between the corresponding elements of the left- and right-hand sides of (8).

Estimate (8) makes it possible to prove, by the method of successive approximations, the existence of a unique solution, analytic in λ in the domain R_δ , of the integral equation (6), representable in the form

$$\mu(z, \lambda) = \tilde{\psi}(z, \lambda) + \int_{\Gamma} R(z, y, \lambda) \tilde{\psi}(y, \lambda) d\Gamma_y, \quad (9)$$

where $R(z, y, \lambda)$ is the resolvent of the kernel $K(z, y, \lambda)$, for which

$$R(z, y, \lambda) = K(z, y, \lambda) + \sum_{n=2}^{\infty} K_n(z, y, \lambda), \quad (10)$$

K_n are the iterations of the kernel K . Moreover, according to (8), from (10) and (9) we obtain the estimates

$$|R(z, y, \lambda)| \leq \frac{CB}{|z - y|^{2-\alpha}} \exp(-\varepsilon|\lambda||z - y|), \quad (11)$$

$$|\mu(z, \lambda)| \leq q, \quad (12)$$

valid in the domain R_δ , where q is a constant column vector of size m .

Thus we prove

Theorem 1. *Under conditions 1°–3° and for sufficiently large R , there exists a solution $u^{(1)}(x, \lambda)$, analytic in λ in the domain R_δ , of problem (1), (2)*

in the case $\Phi(x) \equiv 0$ in D , represented in the form of the simple-layer potential (3), where $P(x, y, \lambda)$ is the fundamental matrix of solutions with singularity at the point $x = y$. The integral equation (6), obtained for the density $\mu(z, \lambda)$ of the potential (3), has, analytic in λ in the domain R_δ , a resolvent $R(z, y, \lambda)$, for which the estimate (11) in R_δ is valid. If D_1 is a domain lying, together with its boundary, in the domain D , then for $u^{(1)}(x, \lambda)$ the inequalities

$$\left| \frac{du^{(1)}(x, \lambda)}{dn_z} \right| \leq q \quad \text{for } x \in D + \Gamma, \quad (13)$$

$$\left| \frac{\partial^k u^{(1)}(x, \lambda)}{\partial x_i^k} \right| \leq \frac{q}{\sigma^{k+1}} \exp(-\varepsilon|\lambda|\sigma), \quad (14)$$

hold, where σ is the distance between the boundaries of the domains D, D_1 .

Under conditions 1°–3°, the existence is also proved of the Green matrix

$$G(x, \xi, \lambda) = P(x, \xi, \lambda) - a(x, \xi, \lambda) \quad (15)$$

of problem (1), (2) (for $\psi(z, \lambda) \equiv 0$ on Γ), by means of which the solution of problem (1), (2) for $\psi(z, \lambda) \equiv 0$ on Γ is represented in the form

$$u^{(2)}(x, \lambda, \Phi) = - \int_D G(x, \xi, \lambda) \Phi(\xi) dD_\xi, \quad (16)$$

where $Q(x, \xi, \lambda)$ is the regular part of the Green matrix, which is also sought in the form of a simple-layer potential

$$Q(x, \xi, \lambda) = \int_\Gamma P(x, y, \lambda) \mu(y, \xi, \lambda) d\Gamma_y, \quad x, \xi \in D. \quad (17)$$

By the scheme of the proof of Theorem 1, one also proves

Theorem 2. *Under the conditions of Theorem 1, for every continuously differentiable and bounded vector-function $\Phi(x)$ in the domain $D + \Gamma$, there exists a solution $u^{(2)}(x, \lambda, \Phi)$ of problem (1), (2) with $\psi(z, \lambda) \equiv 0$ on Γ , defined by formula (16) and analytic in λ in the domain R_δ .*

The regular part $Q(x, \xi, \lambda)$ of the Green matrix $G(x, \xi, \lambda)$, defined by formula (15), is represented in the form of the simple-layer potential (17). For every pair of points $x, \xi \in D_1$ the inequality

$$|\partial^k Q(x, \xi, \lambda)/\partial x_i^k| \leq CB \exp[-\varepsilon|\lambda||x - \xi|]/\sigma^{k+3} \quad (k = 0, 1, 2). \quad (18)$$

holds. For all $x \in D + \Gamma$, $\xi \in D_1$, the estimate

$$|dQ(x, \xi, \lambda)/dn_z| \leq CB \exp[-\varepsilon|\lambda||x - \xi|]/\sigma^2, \quad z \in \Gamma.$$

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CITED LITERATURE

¹ M. L. Rasulov, *The Method of the Contour Integral*, "Nauka," 1964. ² M. L. Rasulov, DAN, 192, No. 6 (1970).

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