ON SEMIGROUPS HAVING SINGULARITIES AT ZERO SUMMABLE WITH A POWER WEIGHT

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Abstract

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ON SEMIGROUPS HAVING SINGULARITIES AT ZERO SUMMABLE WITH A POWER WEIGHT

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As is known, semigroups of linear continuous operators, strongly continuous only for $t > 0$, may have arbitrary growth as $t \to 0$. In this note an attempt is made to extend some results of the theory of semigroups of class C_0 to the classes $L^{(n)}$ of semigroups $T(t)$, strongly continuous for $t > 0$, for which the functions $t^n\|T(t)x\|$ $(x \in X)$ have summable singularities at zero. Semigroups of this type were first considered by Da Prato ([2]). Namely, he considered semigroups $T(t)$ satisfying the estimate $t^\alpha\|T(t)\| \le C$ for some integer α. The results of the present paper make it possible to carry out a finer classification of semigroups, taking into account the case of a fractional exponent α. Moreover, the results obtained make it possible to consider, besides semigroups for which the functions $t^\alpha\|T(t)x\|$ $(x \in X)$ are bounded, also semigroups for which the functions $t^\alpha\|T(t)x\|$ are summable. The principal result of the note is the establishment of a correspondence between semigroups of class $L^{(n)}$ and their infinitesimal operators.

1. Suppose that $T(t)$ is a semigroup of linear continuous operators in a Banach space X, strongly continuous for $t > 0$, ω_0 is its type, and A_0 is its infinitesimal operator, i.e. the operator defined by the equality

$$A_0x = \lim_{t\to 0}\frac{T(t)x - x}{t} \tag{1}$$

on the set $D(A_0)$ of all those $x \in X$ for which the limit (1) exists. Introduce the notation

$$N_0 = \bigcap_{t>0}\{x : T(t)x = 0\}; \qquad X_0 = \overline{\bigcup_{t>0}\{T(t)x : x \in X\}}.$$

Theorem 1. *Let $N_0 \cap X_0 = \{0\}$. Then the infinitesimal operator A_0 admits a closure.*

One of the closed extensions of the operator A_0 can be specified in the following way. Define $D(B)$ as the set of all $x \in X$ for which there exists such a $z \in X_0$ that, for all t and s, $0 < s < t$,

$$T(t)x - T(s)x = \int_s^t T(\tau)z\,d\tau.$$

If the assumptions of Theorem 1 are satisfied, then the element z is determined uniquely by the element x. Therefore the equality $Bx = z$ defines some operator B. This operator is closed and is an extension of the operator A_0.

Next introduce the operator $R(\lambda)$, defined for $\operatorname{Re}\lambda > \omega_0$ by the formula

$$R(\lambda)x = \lim_{t\to 0}\int_t^\infty e^{-\lambda s}T(s)x\,ds \tag{2}$$

on the set D_R of all those $x \in X$ for which the limit (2) exists for arbitrary λ, $\operatorname{Re}\lambda > \omega_0$.

Lemma 1. The operator $R(\lambda)$ maps D_R into D_R for every λ. Moreover, the following relations hold:

a) $R(\lambda)x - R(\mu)x = (\mu - \lambda)R(\lambda)R(\mu)x \quad (x \in D_R)$;

b) $(\lambda I - A_0)R(\lambda)x = x$, if

$$\lim_{t\to 0}\frac{1}{t}\int_0^t T(s)x\,ds = x;$$

c) $R(\lambda)(\lambda I - A_0)x = x \quad (x \in D(A_0))$;

d) $R(\lambda)(\lambda I - B)x = x$, if $x \in D(B)$ and $\lim_{t\to 0} T(t)x = x$;

e) $(\lambda I - B)R(\lambda)x = x \quad (x \in D_R \cap X_0)$.

Lemma 2. If $x \in D_R$, then for any nonnegative integer m the equality

$$R^{m+1}(\lambda)x = \frac{1}{m!}\int_0^\infty t^m e^{-\lambda t}T(t)x\,dt \tag{3}$$

holds.

Let us note that, in contrast to the case of a semigroup of class L ([3]), the operator $R(\lambda)$ need not be bounded.

We shall say that the semigroup $T(t)$ belongs to the class $L^{(n)}$ if $N_0 = \{0\}$, $X_0 = X$, and if for every $x \in X$ the function $t^n\|T(t)x\|$ is summable on every interval of the form $(0, a)$, $a < \infty$.

Lemma 3. Let $T(t)$ be a semigroup of class $L^{(n)}$. Then the operator $S_n(\lambda)$, defined on X by the formula

$$S_n(\lambda)x = \frac{1}{n!}\int_0^\infty t^n e^{-\lambda t}T(t)x\,dt \quad (\operatorname{Re}\lambda > \omega_0), \tag{4}$$

is continuous for $\operatorname{Re}\lambda > \omega_0$, and for every $\omega > \omega_0$ the estimate $\|S_n(\lambda)\| \le M_\omega$ holds $(\operatorname{Re}\lambda > \omega)$.

Comparison of formulas (3) and (4) shows that for semigroups of class $L^{(n)}$ the operator $R^{n+1}(\lambda)$, defined on D_R, is bounded, and the operator $S_n(\lambda)$ is its continuous extension to all of X.

For any $x \in X$ the function $S_n(\lambda)x$ is the Laplace transform of the function $t^n(n!)^{-1}T(t)x$, continuous for $t > 0$. Therefore the usual theorems on the Laplace transform are applicable, in particular the uniqueness theorem. In the case of a semigroup of class $L^{(n)}$, it follows from this theorem that $S_n(\lambda)x = 0$ $(\operatorname{Re}\lambda > \omega_0)$ if and only if $x = 0$. The same can be said about $R(\lambda)$: $R(\lambda)x = 0$ $(x \in D_R)$ if and only if $x = 0$.

2. Let A be a closable operator, and suppose that $D(A^m)$ is dense in X for every m. An analytic function $S_n(\lambda, A)$, defined in some domain $\rho_n(A)$ of the complex plane and taking values in the space of linear continuous operators, will be called the resolvent of order n of the operator A, if from $S_n(\lambda, A)x = 0$ $(\lambda \in \rho_n(A))$ it follows that $x = 0$, and if for $\lambda \in \rho_n(A)$

$$S_n(\lambda, A)Ax = AS_n(\lambda, A)x \quad (x \in D(A));$$

$$S_n(\lambda, A)(\lambda I - A)^{n+1}x = x \quad (x \in D(A^{n+1})). \tag{5}$$

We shall say that the operator A belongs to the class $L_\omega^{(n)}$ if $\rho_n(A) \supseteq \{\lambda : \operatorname{Re}\lambda > \omega\}$, with $\|S_n(\lambda, A)\| \le M$ $(\operatorname{Re}\lambda > \omega)$, and if there exist a nonnegative function $\varphi(t, x)$ $(x \in X,\ t > 0)$, continuous jointly in its variables, and a nonnegative function $\varphi(t)$, bounded on every interval of the form (a, b), $0 < a < b$, for which

$$\lim_{t\to\infty} t^{-1}\ln\varphi(t) < \omega,$$

that

$$\varphi(t, x) \leqslant \varphi(t)\|x\|; \qquad \int_0^\infty t^n\varphi(t, x)e^{-\omega t}\,dt < \infty, \tag{6}$$

$$\|S_n^{(m)}(\tau, A)x\| \leqslant \frac{1}{n!}\int_0^\infty e^{-\tau t}t^{n+m}\varphi(t, x)\,dt \quad (\tau > \omega,\ m = 0, 1, \ldots). \tag{7}$$

Theorem 2. *The infinitesimal operator A_0 of a semigroup $T(t)$ of class $L^{(n)}$ belongs to the class $L_\omega^{(n)}$ for every $\omega > \omega_0$, and $S_n(\lambda, A_0) = S_n(\lambda)$ $(\operatorname{Re}\lambda \geqslant \omega)$.*

Theorem 3. *If A is an operator of class $L_\omega^{(n)}$, then there exists a unique semigroup $T(t)$ of class $L^{(n)}$ whose infinitesimal operator A_0 satisfies the relation $S_n(\lambda, A_0) = S_n(\lambda, A)$. Moreover $\omega_0 \leqslant \omega$, $\|T(t)x\| \leqslant \varphi(t,x)$, and $\|T(t)\| \leqslant \varphi(t)$.*

Let the operator A have, for $\operatorname{Re}\lambda \geqslant \omega$, a resolvent $S_n(\lambda, A)$ of order n, and let there exist a family $\{T(t) : t > 0\}$ of linear continuous operators, strongly continuous in t, such that

$$\int_0^\infty t^n e^{-\omega_1 t}\|T(t)x\|\,dt < \infty$$

for some n and ω_1 and for every x. Suppose, in addition, that

$$S_n(\lambda, A)x = \frac{1}{n!}\int_0^\infty t^n e^{-\lambda t} T(t)x\,dt \quad (\operatorname{Re}\lambda > \max(\omega, \omega_1)).$$

Then it follows from Theorem 3 that $T(t)$ is a semigroup of class $L^{(n)}$.

If we fix the functions $\varphi(t,x)$ and $\varphi(t)$, then the estimates (7) give us a necessary and sufficient condition for the function $S_n(\tau, A)$ to correspond to a semigroup $T(t)$ satisfying the estimates $\|T(t)x\| \leqslant \varphi(t,x)$ and $\|T(t)\| \leqslant \varphi(t)$. By choosing various functions $\varphi(t,x)$ and $\varphi(t)$, we shall obtain conditions for a semigroup to belong to one or another class of semigroups. In the case $n = 0$ this path is considered in detail in ([3]).

It is natural to supplement Theorem 3 with the following assertion.

Theorem 4. *Let a closed operator A belong to the class $L_\omega^{(n)}$ and have a resolvent $R(\lambda, A)$ for $\operatorname{Re}\lambda > \omega$. Then the closure $\overline{A}_0$ of the operator A_0 coincides with the operator A, and $S_n(\lambda, A) = R^{n+1}(\lambda, A)$.*

3. We shall call a semigroup $T(t)$ of class $L^{(n)}$ a semigroup of class $L_0^{(n)}$ if the function $t^n\|T(t)\|$ is summable on every interval of the form $(0, a)$, $a < \infty$. Obviously, if $T(t)$ is a semigroup of class $L_0^{(n)}$, then the estimates

$$\|S_n^{(m)}(\lambda)\| \leqslant \frac{1}{n!}\int_0^\infty t^{n+m} e^{-t\operatorname{Re}\lambda}\|T(t)\|\,dt \leqslant \frac{1}{n!}\int_0^\infty t^{n+m} e^{-t\operatorname{Re}\lambda}\varphi(t)\,dt,$$

are valid if $\|T(t)\| \leqslant \varphi(t)$ and the function $t^n\varphi(t)e^{-\omega t}$ is summable on $(0, \infty)$. The converse assertion is also valid:

Theorem 5. *Let an operator A have a resolvent $S_n(\lambda, A)$ of order n for real $\tau > \omega$, and suppose the estimates*

$$\|S_n^{(m)}(\tau, A)\| \leqslant \frac{1}{n!}\int_0^\infty t^{n+m}e^{-\tau t}\varphi(t)\,dt \quad (\tau > \omega,\ m = 0, 1, ...), \tag{8}$$

where the function $\varphi(t)$ is nonnegative and $t^n\varphi(t)e^{-\omega t}$ is summable on $(0, \infty)$. Then there exists a semigroup $T(t)$ of class $L_0^{(n)}$ for which $S_n(\tau) = S_n(\tau, A)$ for $\tau > 0$.

4. As an application, let us consider semigroups $T(t)$ with $N_0 = \{0\}$ and $X_0 = X$, satisfying the estimate $\|T(t)\| \leq Ct^{-\alpha}e^{\omega t}$. We shall call these semigroups semigroups of class C_α. They belong to the classes $L_0^{(n)}$ for $n \geq [\alpha]$. The estimates (8) take for them a simpler form.

Theorem 6. Let $T(t)$ be a semigroup of class C_α. Then its infinitesimal operator A_0 has, for $\operatorname{Re}\lambda > \omega$, a resolvent $S_n(\lambda, A_0)$ of order n ($n \geq [\alpha]$), and

$$\left\|S_n^{(m)}(\lambda, A_0)\right\| \leq \frac{C\Gamma(m+n+1-\alpha)}{n!(\operatorname{Re}\lambda - \omega)^{m+n+1-\alpha}} \qquad (\operatorname{Re}\lambda > \omega,\ m = 0, 1, ...). \tag{9}$$

Conversely, suppose an operator A has, for real $\tau > \omega$, a resolvent $S_n(\tau, A)$ of order n, and the estimate

$$\left\|S_n^{(m)}(\tau, A)\right\| \leq \frac{C\Gamma(m+n+1-\alpha)}{n!(\tau - \omega)^{m+n+1-\alpha}} \qquad (\tau > \omega,\ m = 0, 1, ...) \tag{10}$$

holds. Then there exists a unique semigroup $T(t)$ of class $L_0^{(n)}$, satisfying the estimate $\|T(t)\| \leq Ct^{-\alpha}e^{\omega t}$, for which

$$S_n(\tau, A)x = \frac{1}{n!}\int_0^\infty t^n e^{-\tau t}T(t)x\,dt.$$

Theorem 6 is a generalization of the Da Prato theorem ([2]) to the case of fractional α. Another theorem of a similar kind for semigroups satisfying the estimate $\|T(t)\| \leq Ct^{-\alpha}e^{\omega t}$ was proved by P. E. Sobolevskii ([4]).

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Note: Figure translations are in progress. See original paper for figures.