# ON ONE REPRESENTATION OF THE BOLTZMANN EQUATION

AERODYNAMICS

1970

SovietRxiv

## Abstract

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UDC 533.7

*AERODYNAMICS*

**I. A. ENDER, A. Ya. ENDER**

# ON ONE REPRESENTATION OF THE BOLTZMANN EQUATION

*(Presented by Academician G. I. Petrov on 24 XII 1969)*

The main difficulty in solving the Boltzmann equation lies in the complexity of the collision integral. We shall show, first on the example of a simple kinetic problem, how it can be transformed into a simpler and more transparent form.

We shall consider a homogeneous gas whose distribution function at the initial instant depends only on the magnitude of the velocity, i.e., is spherically symmetric. In this case it can be represented in the form of an integral of Maxwellian distributions with arbitrary temperatures, i.e.,

$$f(v,t) = \int_0^\infty M(\alpha, v)\,\varphi(\alpha, t)\,d\alpha, \tag{1}$$

where $M(\alpha, v) = (\alpha/\pi)^{3/2} e^{-\alpha v^2}$, $\alpha = m/2kT$, and $\varphi(\alpha, t)$ is the weight of the corresponding Maxwellian. Relation (1) reduces to the Laplace transform from the $v^2$-space of images to the $\alpha$-space of originals and is admissible for a sufficiently broad class of spherically symmetric distribution functions.

In the case under consideration the Boltzmann equation has the form

$$\partial f/\partial t = nI_{\text{ct}}(f, f'); \tag{2}$$

$I_{\text{ct}}(f, f')$ is the integral collision operator. Substituting (1) into the right-hand side of (2) and changing the order of integration, we obtain:

$$\frac{\partial f}{\partial t} = n\int_0^\infty \int_0^\infty \varphi(\alpha_1, t)\varphi(\alpha_2, t) I_{\text{ct}}^M(\alpha_1, \alpha_2, v)\,d\alpha_1 d\alpha_2; \tag{3}$$

$I_{\text{ct}}^M(\alpha_1, \alpha_2, v)$ is the collision integral of two Maxwellians with inverse temperatures $\alpha_1$ and $\alpha_2$. Expanding both sides of equation (3) in Maxwellians, we obtain the Boltzmann equation in $\alpha$-space:

$$\frac{\partial\varphi(\alpha,t)}{\partial t} = n\int_0^\infty\int_0^\infty A(\alpha,\alpha_1,\alpha_2)\,\varphi(\alpha_1,t)\varphi(\alpha_2,t)\,d\alpha_1 d\alpha_2. \tag{4}$$

The kernel $A(\alpha,\alpha_1,\alpha_2)$ is the coefficient in the expansion with respect to $M(\alpha,v)$ of the collision integral of two Maxwellians with temperatures $T_1$ and $T_2$:

$$I^M_{\mathrm{ct}}(\alpha_1,\alpha_2,v) = \int_0^\infty M(\alpha,v)\,A(\alpha,\alpha_1,\alpha_2)\,d\alpha. \tag{5}$$

To find $A(\alpha,\alpha_1,\alpha_2)$, the well-developed apparatus of Laplace transforms may be used. The advantage of writing the Boltzmann equation in the form (5) is connected with a significant simplification of the collision integral, since the kernel is computed analytically and is standard for an entire class of problems.

For the model of "hard elastic spheres" we have:

$$A(\alpha,\alpha_1,\alpha_2) = \overline{\mathrm{Rel}}_+ - \overline{\mathrm{Rel}}_-,$$

$$\overline{\mathrm{Rel}}_+ = \left(\frac{\alpha_1\alpha_2}{\pi^2}\right)^{3/2}\frac{\sigma\pi^{5/2}n}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\frac{U(\alpha_1+\alpha_2-\alpha)}{\alpha^{3/3}}\left[\frac{\sqrt{\alpha_1}\,U(\alpha-\alpha_2)}{\sqrt{\alpha-\alpha_2}} - \frac{\sqrt{\alpha_2}\,U(\alpha-\alpha_1)}{\sqrt{\alpha-\alpha_1}}\right], \tag{6}$$

$$\overline{\mathrm{Rel}}_- = \left(\frac{\alpha_1\alpha_2}{\pi^2}\right)^{3/2}\frac{\sigma\pi^{5/2}n}{\alpha_2^2}\frac{U(\alpha_1+\alpha_2-\alpha)}{\alpha^{3/2}}\left[\frac{U(\alpha-\alpha_1)}{2\sqrt{\alpha_2}\sqrt{\alpha-\alpha_1}} + \sqrt{\alpha_2}\frac{d}{d\alpha}\frac{U(\alpha-\alpha_1)}{\sqrt{\alpha-\alpha_1}}\right]. \tag{7}$$

Here $U(x)$ is the Heaviside step function, and $\sigma$ is the total scattering cross section.

From formulas (6), (7), the boundedness of the domain of definition of the kernel is immediately evident, i.e. $A(\alpha,\alpha_1,\alpha_2)=0$ for $\alpha<\min(\alpha_1,\alpha_2)$ and $\alpha>\alpha_1+\alpha_2$. Thus, a Maxwellian with a temperature greater than the maximum of the initial ones cannot appear in the spectrum of the distribution function. Therefore, if all temperature values are referred to this temperature, then $\varphi(T)$ will be nonzero only on the interval $(0,1)$.

Passing to the numerical formulation of the problem, we divide the entire interval $(0,1)$ into $N$ equal parts and, on each interval $(x_k,x_{k+1})$, instead of the function $\varphi(T)$ we shall consider

$$a_k = \int_{x_k}^{x_{k+1}} \varphi(x)\,dx. \tag{8}$$

Figure 1

Figure 1: Figure 1

Integrating both sides of equation (5) over the interval $(x_k, x_{k+1})$ and substituting into the right-hand side $\varphi = \sum_k a_k \delta(x - x_k^*)$ $(x_k < x_k^* < x_{k+1})$, we obtain

$$\frac{da_k(t)}{dt} = \sum_{l,p} A_{k,l,p} a_l(t) a_p(t). \tag{9}$$

The solution of this system of nonlinear ordinary differential equations was carried out on the BESM-4. In the case where the initial distribution function consists of two Maxwellians* with the initial temperature ratio $T_2/T_1 = 5$, the solution is shown in Fig. 1. Here $\varphi(T)$ is depicted at different moments of time. The characteristic time was chosen as $t = n\sigma(2kT_0/m)^{1/2}$, where $T_0$ is the maximum of the initial temperatures. Fig. 2 gives the dependence of $f(v)$ at the corresponding moments of time. $f(v)$ is computed by the formula

$$f(v) = \sum_k a_k M(\alpha_k, v). \tag{10}$$

After the numerical study one may say that, in order to construct $f(v,t)$ at one time step with an accuracy of up to the 3rd–4th significant digits, about 20 sec on the BESM-4 is required, and about 10 min to study the complete evolution of the system. Such accuracy with minimal time expenditure is due to the fact that a significant part of the calculation (the computation of the kernel) has been performed analytically. It can be shown that, in spherically symmetric problems, the form of the kernel becomes only slightly more complicated for other scattering cross sections as well.

In the case where the distribution function is not spherically symmetric, one may use the more general expansion:

$$f(\mathbf{v}, \mathbf{r}, t) = \int_{-\infty}^{+\infty} \int_0^{\infty} M(\alpha, \mathbf{v}, \mathbf{u}) \varphi(\alpha, \mathbf{u}, \mathbf{r}, t)\, d\alpha\, d\mathbf{u}. \tag{11}$$

Here

$$M(\alpha, \mathbf{v}, \mathbf{u}) = (\alpha/\pi)^{3/2} e^{-\alpha(\mathbf{v}-\mathbf{u})^2}, \qquad \alpha = m/2kT.$$

* An analogous problem, with direct numerical integration of the Boltzmann equation, was solved in (1).

Figure 2

Figure 2: Figure 2

**Fig. 1.** Distribution function $\varphi(T)$ at different moments of time for $N = 30$

**Fig. 2.** Distribution function $f'(v')$ $\left(v' = \sqrt{a_0}\, v,\ f'(v') = \frac{1}{a_0^{3/2}} f(v')\right)$. The dashed curve corresponds to equilibrium

Such a representation was first proposed in ([2]), but there it was reduced to the solution of moment equations. Substituting (11) into the collision integral and carrying out transformations analogous to (3), (4), (5), we obtain

$$\frac{\partial\varphi(a,\mathbf{u},t)}{\partial t} = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} A(a,\mathbf{u},a_1,\mathbf{u}_1,a_2,\mathbf{u}_2)\varphi(a_1,\mathbf{u}_1,t)\,\varphi(a_2,\mathbf{u}_2,t)\times$$

$$\times\, da_1\, d\mathbf{u}_1\, da_2\, d\mathbf{u}_2. \tag{12}$$

Here $A(a,\mathbf{u},a_1,\mathbf{u}_1,a_2,\mathbf{u}_2)$ is the coefficient in the expansion, in $M(a,\mathbf{v},\mathbf{u})$, of the collision integral of two Maxwellians with different temperatures

$(\alpha_1,\alpha_2)$ and velocities $(\mathbf{u}_1,\mathbf{u}_2)$. Such a decomposition is carried out uniquely and again reduces to a twice-repeated Laplace transformation, if the vectors $\mathbf{u}-\mathbf{u}_1$ are chosen parallel to the vector $\mathbf{u}_2-\mathbf{u}_1$.

It can also be shown that, if the problem is spatially inhomogeneous, then the Boltzmann equation in the $\alpha$—$\mathbf{u}$-representation has the form:

$$\frac{\partial\varphi}{\partial t} + \mathbf{u}\cdot\frac{\partial\varphi}{\partial\mathbf{r}} - \frac{1}{2\alpha}\frac{\partial^2\varphi}{\partial\mathbf{u}\,\partial\mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial\varphi}{\partial\mathbf{u}} = \tag{13}$$

$$= \int_{-\infty}^{+\infty}\int_{0}^{\infty}\int_{-\infty}^{+\infty}\int_{0}^{\infty} A(\alpha,\mathbf{u},\alpha_1,\mathbf{u}_1,\alpha_2,\mathbf{u}_2)\times\varphi(\alpha_1,\mathbf{u}_1,\mathbf{r},t)\varphi(\alpha_2,\mathbf{u}_2,\mathbf{r},t)\,d\alpha_1\,d\mathbf{u}_1\,d\alpha_2\,d\mathbf{u}_2.$$

Here $\partial^2\varphi/\partial\mathbf{u}\,\partial\mathbf{r} = \partial^2\varphi/\partial u_x\partial x + \partial^2\varphi/\partial u_y\partial y + \partial^2\varphi/\partial u_z\partial z$.

Physico-Technical Institute named after A. F. Ioffe
Academy of Sciences of the USSR
Leningrad

Received
10 XII 1969

## CITED LITERATURE

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[2] F. Weitzsch, Ann. Phys., 7, 403 (1961).

*Note: Figure translations are in progress. See original paper for figures.*