DESCRIPTION OF RELATIVELY PROJECTIVE IDEALS IN THE ALGEBRAS \(C(\Omega)\)

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Abstract

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DESCRIPTION OF RELATIVELY PROJECTIVE IDEALS IN THE ALGEBRAS $C(\Omega)$

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1. Let A be a Banach algebra with identity; by an A-module we shall always mean a left Banach module over A. An A-module is called relatively projective (in what follows, for brevity, projective) if it is representable as a direct summand of a module of the form $A\widehat{\otimes}X$, where X is an arbitrary Banach space, $\widehat{\otimes}$ denotes the tensor product with the greatest cross-norm ([1]), and the external multiplication is given by the formula $a\cdot(b\otimes x) = ab\otimes x$; $a, b \in A$, $x \in X$. The algebra A is called relatively hereditary (hereafter hereditary) if every closed left ideal in A is a projective A-module.

As was shown in ([2]), the question of projectivity of ideals in a given Banach algebra is closely connected with the construction of an important class of its singular extensions. It was also found there that already in considering the algebras $C(\Omega)$ (consisting of all continuous functions on a compact space Ω) not all of them turn out to be hereditary.

In the present work a complete description is given of the projective ideals in the algebras $C(\Omega)$, and thereby of the hereditary algebras in this class.

The space of modular maximal ideals of a commutative Banach algebra I (generally speaking, without identity) will be called its spectrum; recall ([3]) that the spectrum of a closed ideal I in a commutative Banach algebra A with spectrum Ω is, up to homeomorphism, the open set

$$\Omega_0 = \{t \in \Omega;\ f(t) \neq 0 \text{ for some } f \in I\}.$$

The main result of the work is

Theorem 4. *A closed ideal $I \subset C(\Omega)$ is a projective $C(\Omega)$-module if and only if its spectrum is paracompact.*

An automatic consequence is

Theorem 5. *The algebra $C(\Omega)$ is hereditary if and only if every open set in Ω is paracompact.*

As is known, the spectra of maximal ideals in $C(\Omega)$ are sets of the form $\Omega \setminus \{t\};\ t \in \Omega$. Therefore, combining Theorem 4 with the results of ([2]), we obtain the following assertion.

Theorem 6. *The following properties of the compact space Ω are equivalent:*

(I) $s.\dim C(\Omega) \leq 1$;

(II) every semiscalar extension of the algebra $C(\Omega)$ is strongly decomposable;

(III) every set of the form $\Omega \setminus \{t\};\ t \in \Omega$ is paracompact.

Passing to the proof of Theorem 4, we note that both the sufficiency and the necessity follow from assertions of a more general character, which are applicable also to other Banach algebras besides $C(\Omega)$.

2. **Proof of sufficiency.**

For an arbitrary (index) set Λ, by $N(\Lambda)$ we shall denote the set of finite subsets of Λ, ordered by inclusion. A net $e_\lambda;\ \lambda \in N(\Lambda)$ of elements of the Banach algebra I will be called a right approximate identity (r.a.i.) in I if for every $a \in I$, $\lim_\lambda ae_\lambda = a$.

Theorem 1. *Let I be a closed left ideal in a Banach algebra A. Suppose, further, that there exists a bounded set $g_\mu \in I;\ \mu \in \Lambda$ with the following properties:*

(I) for any $\lambda = (\mu_1, \dots, \mu_n) \in N(\Lambda)$, $a \in A$, and any root ξ of the n-th degree from unity,

$$\left\| \sum_{i=1}^{n} \xi^i a g_{\mu_i} \right\| \leq C \max_i \|a g_{\mu_i}\|,$$

where C is some constant;

(II) the net e_λ, $\lambda \in N(\Lambda)$, where for $\lambda = (\mu_1, \dots, \mu_n)$

$$e_\lambda = \sum_{i=1}^{n} g_{\mu_i}^2$$

is an approximate identity in I;

(III) for any $x \in I$ and $\varepsilon > 0$, $\|x g_\mu\| \geq \varepsilon$ only for a finite set of μ in Λ.

Then I is a projective A-module.

Consider the module $A \widehat{\otimes} I$ and the epimorphism $\pi : A \widehat{\otimes} I \to I$ such that $\pi(a \otimes x) = ax$, $a \in A$, $x \in I$. Obviously, it suffices to construct a morphism of A-modules $\rho : I \to A \widehat{\otimes} I$ such that $\pi \circ \rho = 1_I$.

Lemma 1.1. For any $x \in I$, $\lambda = (\mu_1, \dots, \mu_n) \in N(\Lambda)$, and $u \in A \widehat{\otimes} I$:

$$u = \sum_{i=1}^{n} x g_{\mu_i} \otimes g_{\mu_i}$$

the inequality

$$\|u\| \leq C \max_i \|x g_{\mu_i}\| \cdot \max_i \|g_{\mu_i}\|$$

holds.

As is easy to verify, for a primitive root ξ of the n-th degree from unity,

$$u = \frac{1}{n} \sum_{i=1}^{n} \left(x g_{\mu_1} + \xi^i x g_{\mu_2} + \cdots + \xi^{i(n-1)} x g_{\mu_n} \right) \otimes \left(g_{\mu_1} + \xi^{-i} g_{\mu_2} + \cdots + \xi^{-i(n-1)} g_{\mu_n} \right).$$

Consequently, by the definition of the norm in $A \widehat{\otimes} I$, $\|u\|$ does not exceed the arithmetic mean of the products of the norms of the tensor "factors." Thus the required inequality follows from (I).

Fix $x \in I$ and, for any $\lambda = (\mu_1, \dots, \mu_n) \in N(\Lambda)$, put

$$u_\lambda = \sum_{i=1}^{n} x g_{\mu_i} \otimes g_{\mu_i}.$$

Lemma 1.2. The net u_λ converges in $A \widehat{\otimes} I$.

In view of the completeness of $A \widehat{\otimes} I$, it suffices to show that the net u_λ is fundamental. Let $\varepsilon > 0$; take a finite set $\overline{\lambda} \in N(\Lambda)$ such that, for $\mu \in \overline{\lambda}$, $\|x g_\mu\| < \varepsilon$ (such a set exists by (III)). Then for $\lambda > \overline{\lambda}$, λ', λ'' the difference $u_{\lambda'} - u_{\lambda''}$ is representable in the form

$$\sum_{i=1}^{m} x g_{\mu_i} \otimes g_{\mu_i} - \sum_{j=1}^{n} x g_{\nu_j} \otimes g_{\nu_j},$$

where $\mu_i, \nu_j \in \lambda$. Hence, using Lemma 1.1, we obtain that

$$\|u_{\lambda'} - u_{\lambda''}\| \leq 2C\varepsilon \max_{\mu \in \Lambda} \|g_\mu\|;$$

thereby the lemma is proved.

End of the proof. Define the mapping $\rho : I \to A \widehat{\otimes} I$ by setting, for $x \in I$,

$$\rho(x) = \lim_{\lambda} \left(\sum_{\mu \in \lambda} x g_\mu \otimes g_\mu \right) \in A \widehat{\otimes} I.$$

Obviously, ρ is a morphism of (Banach) A-modules. Further, as is easy to see, $\pi\rho(x) = x$ for any $x \in I$. Thus, the proof of Theorem 1 is complete.

Theorem 2. Let Ω be a compact space, and let Ω_0 be its open paracompact subset. Then in the ideal

$$I = \{f \in C(\Omega) : \ f(t) = 0 \text{ for } t \in \Omega \setminus \Omega_0\}$$

there exists a set $\{g_\mu\}$, $\mu \in \Lambda$, satisfying conditions (I)–(III) of Theorem 1.

The main auxiliary proposition is the following lemma, suggested to the author by A. V. Arkhangel' skii.

Lemma 2.1. For every paracompact locally compact topological space Ω_0 there exists an open covering $\mathfrak{U}$ by relatively compact sets such that every point of Ω_0 has a neighborhood intersecting no more than three sets from $\mathfrak{U}$.

Take any open covering of the space Ω_0 by relatively compact sets and inscribe in it a locally finite open covering $\mathfrak{B}$. For any point $t \in \Omega_0$ define, by induction, the sets $S_n(t)$, $n = 0, 1, \dots$, putting $S_0(t) = \{t\}$ (the one-point set) and

$$S_n(t) = \bigcup \{U \in \mathfrak{B} : \ U \cap S_{n-1}(t) = \emptyset\}$$

for $n > 0$; put further

$$S(t) = \bigcup_{n=1}^{\infty} S_n(t).$$

Obviously, for every $t \in \Omega_0$, $S(t)$ is a σ-compact set, and for distinct t these sets either coincide or do not intersect. Using the axiom of choice, take $T \subset \Omega_0$ such that for $t', t'' \in T$, $t' \neq t''$,

$$S(t') \cap S(t'') = \emptyset$$

and

$$\bigcup_{t \in T} S(t) = \Omega_0.$$

Now define, for $t \in T$ and $n > 0$,

$$V_{t,n} = S_{n+1}(t) \setminus \overline{S_{n-1}(t)};$$

put further $V_{t,0} = S_1(t)$ and $\mathfrak{U} = \{V_{t,n};\ t \in T;\ n \geq 0\}$. $\mathfrak{U}$ is an open covering of Ω_0 by relatively compact sets; moreover, as is easy to see, $V_{t',m} \cap V_{t'',n}$ can be nonempty only when $t' = t''$ and $|m-n| \leq 1$. Hence at every point $s \in \Omega_0$, $s \in V_{t,n}$, there is a neighborhood (namely $V_{t,n}$ itself) intersecting no more than three sets from $\mathfrak{U}$. The lemma is proved.

End of the proof. Apply Lemma 2.1 to the spectrum Ω_0 of the ideal I; let $\mathfrak{U}$ be the corresponding covering. Denote by h_μ, $\mu \in \Lambda$, $0 \leq h_\mu \leq 1$, a partition of unity subordinate to $\mathfrak{U}$ (see Kelley [4]); all these functions, obviously, belong to I. Put $g_\mu = \sqrt{h_\mu}$. Then, as is not difficult to see from the definition of the norm in $C(\Omega)$ and the equality $\sum_\mu g_\mu^2(t) = 1$, in which for every $t \in \Omega_0$ no more than three summands are different from zero, it follows that

$$\max_\mu \|g_\mu\| \leq 1$$

and

$$\left\| \sum_{i=1}^{n} \xi^i a g_{\mu_i} \right\| \leq \sqrt{3} \max_i \|a g_{\mu_i}\|$$

for every $a \in C(\Omega)$, i.e. (I) is satisfied. Further, any compact $K \subset \Omega_0$ intersects only a finite number of sets of the (locally finite) covering $\mathfrak{U}$, and for all $x \in I$, $x(t) \to 0$ as $t \to \infty$ in Ω_0; hence (II) and (III) easily follow. Thus Theorem 2 is proved.

3. Proof of necessity

Theorem 3. Let A be a commutative Banach algebra with spectrum Ω, and let I be a closed ideal in A with spectrum $\Omega_0 \subset \Omega$, which is a projective A-module. Then the space Ω_0 is paracompact.

We shall divide the proof into a number of lemmas.

Lemma 3.1. There exists a continuous complex function $F(s,t)$ on the topological product $\Omega_0 \times \Omega$ possessing the properties $F(s,s) = 1$, $s \in \Omega_0$, and $F(s,t) = 0$ for $t \in \Omega_0$.

Let

$$i:\ A \widehat{\otimes} I \to C(\Omega \times \Omega)$$

be the natural embedding of A-modules [2]. For brevity we shall write $u(s,t)$ instead of $i(u)(s,t)$ for $u \in A\widehat{\otimes}I$. As is easy to see, $u(s,t) = 0$ for $t \in \Omega_0$, $u \in A\widehat{\otimes}I$.

Since I is projective, there exists a morphism of A-modules $\rho : I \to A\widehat{\otimes}I$ such that $\pi \circ \rho = 1_I$. For $s \in \Omega_0$ take $x \in I$ such that $x(s) = 1$, and put, for every $t \in \Omega$,

$$F(s,t) = \rho(x)(s,t).$$

As is easy to see, the function $F : \Omega_0 \times \Omega \to \mathbf{C}$ does not depend on the choice of x and is continuous on $\Omega_0 \times \Omega$.

Now take, for any $s \in \Omega_0$, an element $x \in I$ such that $x(s) = 1$; then

$$F(s,s) = \rho(x)(s,s) = [\pi \cdot \rho(x)](s) = x(s) = 1$$

and

$$F(s,t) = \rho(x)(s,t) = 0$$

for $t \in \Omega_0$. The lemma is proved.

Lemma 3.2. Let $G(s,t)$ be a real nonnegative bounded ⋯

continuous function on $\Omega_0 \times \Omega_0$ such that, for every compact set $K \subset \Omega_0$, $G(s,t) \to 0$ uniformly as $s \to \infty$, $t \in K$. Then, for every subset $M \subset \Omega_0$, the function

$$m(t) = \sup_{s \in M} G(s,t)$$

is continuous on Ω_0.

Suppose, on the contrary, that the (lower semicontinuous) function $m(t)$ has a discontinuity at the point $t_0 \in \Omega_0$; then there exist $\varepsilon > 0$ and a net t_λ, $\lambda \in \Lambda$, such that $\lim_\lambda t_\lambda = t_0$ and $m(t_\lambda) > m(t_0) + \varepsilon$ for all $\lambda \in \Lambda$. Therefore there is also a net s_λ, $\lambda \in \Lambda$, $s_\lambda \in M$, such that $G(s_\lambda, t_\lambda) > m(t_0) + \varepsilon$. In view of $\lim_\lambda t_\lambda = t_0 \in \Omega_0$, we may, without loss of generality, assume that all $t_\lambda \in K$ for some compact set $K \subset \Omega_0$. But, by the hypothesis of the lemma, $G(s,t) \to 0$ uniformly as $s \to \infty$, $t \in K$. Therefore the net s_λ, being situated outside a certain neighborhood of infinity, must have a limit point $s_0 \in \Omega_0$; evidently, without loss of generality, we may assume that $s_0 = \lim_\lambda s_\lambda$.

But then $(s_0, t_0) = \lim_\lambda (s_\lambda, t_\lambda)$; consequently, $G(s_0, t_0) = \lim_\lambda G(s_\lambda, t_\lambda)$. Hence $m(t_0) \geq \lim_\lambda G(s_\lambda, t_\lambda)$, which contradicts the inequality $G(s_\lambda, t_\lambda) > m(t_0) + \varepsilon$. This contradiction proves the lemma.

Lemma 3.3. There exists a family $\{f_\mu;\ \mu \in \Lambda\}$ of continuous nonnegative functions on Ω_0, tending to zero at infinity and such that

$$\sum_\mu f_\mu(t) = 1$$

for every $t \in \Omega_0$.

Take the function $F(s,t)$ from Lemma 3.1 and, for each $s, t \in \Omega_0$, set

$$G(s,t) = \min\{|F(s,t)|;\ 1\} \cdot \min\{|F(t,s)|;\ 1\}.$$

Then the function $G(s,t)$ satisfies the conditions of Lemma 3.2 and, moreover,

$$\sup_{s\in\Omega_0} G(s,t) = 1.$$

Arrange the points of the space Ω_0 into a transfinite sequence and define, for an ordinal α, the function

$$f_\alpha(t) = \sup_{\beta\le\alpha} G(s_\beta, t) - \sup_{\beta<\alpha} G(s_\beta, t).$$

It follows from Lemma 3.2 that all f_α are continuous; further, in view of

$$f_\alpha(t) \le G(s_\alpha, t), \qquad \lim_{t\to\infty} f_\alpha(t) = 0.$$

Finally, for any $t \in \Omega_0$,

$$\sum_{\alpha:\ s_\alpha\in\Omega} f_\alpha(t) = \sup_{s\in\Omega_0} G(s,t) = 1.$$

Thus the proof of the lemma is complete.

End of the proof. For an ordinal α and a natural number n, set

$$U_{\alpha,n} = \{s \in \Omega_0 :\ f_\alpha(s) > 1/n\}$$

(f_α is from the preceding lemma) and take $\mathfrak{U}_n = \{U_{\alpha,n} \ne \emptyset\}$; evidently, for each n the system $\mathfrak{U}_n$ is locally finite.

Let now $\mathfrak{B}$ be an arbitrary open cover of the space Ω_0. Since each $U_{\alpha,n}$ is relatively compact, there exists a finite system $\{V_k \in \mathfrak{B}\};\ 1 \le k \le m_{\alpha,n}$, covering $U_{\alpha,n}$. Put $W_{\alpha,n,k} = U_{\alpha,n} \cap V_k$, and for each n take

$$\mathfrak{B}_n = \{W_{\alpha,n,k} \ne \emptyset\}.$$

The system $\mathfrak{B}_n$ is inscribed in $\mathfrak{U}$ and, evidently, is locally finite; furthermore, since each $s \in \Omega_0$ belongs to some $U_{\alpha,n}$, and hence also to some $W_{\alpha,n,k}$, the system

$$\mathfrak{B} = \bigcup_n \mathfrak{B}_n$$

is a cover of the space Ω_0.

Thus, in every open cover of the space Ω_0 one can inscribe an open σ-locally finite cover. Consequently (Kelley [4]), Ω_0 is paracompact, and Theorem 3 is proved.

The theorem obtained contains, as a special case, the "necessary"part of Theorem 4, while Theorems 1 and 2 together give "sufficiency." Thus Theorem 4, and along with it its direct consequences—Theorems 5 and 6—are proved.

In conclusion, the author considers it his pleasant duty to note the assistance which A. V. Arhangel' skii repeatedly rendered him with his advice.

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Note: Figure translations are in progress. See original paper for figures.