ON THE MOTION OF TEST MASSES IN THE GRAVITATIONAL FIELD OF A ROTATING BODY

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Abstract

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ON THE MOTION OF TEST MASSES IN THE GRAVITATIONAL FIELD OF A ROTATING BODY

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As is known, the static gravitational field of a massive rotating body contains components analogous to the components of the magnetic field of a circular current (at large distances), which affect the motion of test bodies, leading to effects first considered by Lense and Thirring ([1]), in particular to a specific precession of the orbit of a test body. These effects are sometimes connected with Mach' s principle (see, for example, ([2])), according to which, when a massive body rotates, the local inertial frame near it must be partially dragged into rotation. In other words, one should expect that, other conditions being equal, the angular velocity of motion of a test body around a rotating mass should be greater (if its vector coincides in direction with the angular momentum vector of this mass) than the angular velocity in the opposite direction. This effect can be associated with the analogue of the Zeeman effect indicated in the theory of gravitation by Zel' dovich ([3]), although in the Zeeman effect one usually deals with an external magnetic field and not with the magnetic field of the nucleus, whereas here the attracting center serves simultaneously as the source of the quasimagnetic gravitational field. This effect should be most significant when the Lense-Thirring frequency is close to the frequency of the zero (Newtonian) approximation; however, in practice it is only a small correction. Thus, in the simplest case of the motion of two test masses toward each other along one and the same circular equatorial orbit, their periods of revolution must be different. In view of this, the point of their meeting must drift in the direction of rotation of the central body. Denoting the angular velocities of motion of the test masses by $\dot{\varphi}_+$ and $\dot{\varphi}_-$ and, for definiteness, taking $\dot{\varphi}_+ > 0 > \dot{\varphi}_-$, one may write the condition for meeting in the form $t(\dot{\varphi}_+ - \dot{\varphi}_-) = 2\pi n$, where n is an integer ($n = 0$ corresponds to the initial meeting, $n = 2$ to the meeting after one revolution); t is the time elapsed from the moment of the initial meeting. Putting $n = 2$, we find the displacement of the meeting point in one revolution:

$$\delta = 4\pi\dot{\varphi}_+/(\dot{\varphi}_+ - \dot{\varphi}_-) - 2\pi = 2\pi(\dot{\varphi}_+ + \dot{\varphi}_-)/(\dot{\varphi}_+ - \dot{\varphi}_-).$$

Hence there follows the effective magnitude of the drift of the meeting point per unit time (in radians)

$$\delta/T = (\dot{\varphi}_+ + \dot{\varphi}_-)/2.$$

We shall proceed here from the exact solution for the external gravitational field of a rotating symmetric mass, obtained by Kerr ([4]) (see also ([5])),

$$ds^2 = \left(1 - \frac{2\gamma mr}{r^2 + a^2\cos^2\theta}\right) dx^{0\,2} - \frac{r^2 + a^2\cos^2\theta}{r^2 - 2\gamma mr + a^2}\, dr^2 - \frac{4\gamma Lr\sin^2\theta}{r^2 + a^2\cos^2\theta}\, d\varphi\, dx^0 - \left(r^2 + a^2 + \frac{2\gamma ma^2 r\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta\, d\varphi^2 - \left(r^2 + a^2\cos^2\theta\right) d\theta^2$$

(the speed of light $c = 1$; m is the mass of the body; L is its angular momentum; $\gamma -$

Newtonian gravitational constant; $am = L$), although in a reasonable approximation the results will, of course, also be valid for the known approximate solution of Einstein' s equations (see, for example, (6)). The radial component of the geodesic equation in the form

$$\frac{d}{ds}\left(g_{\mu\nu}\frac{dx^\nu}{ds}\right) = \frac{1}{2}g_{\alpha\beta,\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$$

under the assumption of a circular equatorial orbit gives

$$g_{00,r}\left(\frac{dx^0}{ds}\right)^2 + g_{\varphi\varphi,r}\left(\frac{d\varphi}{ds}\right)^2 + 2g_{0\varphi,r}\frac{dx^0}{ds}\frac{d\varphi}{ds} = 0.$$

Hence there follow two exact solutions for the angular velocities of motion of the test masses (counter-motion):

$$\left(\frac{d\varphi}{dx^0}\right)_\pm = \frac{-g_{0\varphi,r} \mp \sqrt{(g_{0\varphi,r})^2 - g_{00,r}g_{\varphi\varphi,r}}}{g_{\varphi\varphi,r}}.$$

These angular velocities can be represented in the form

$$\dot{\varphi}_\pm = \frac{\pm\sqrt{\gamma m/r + \gamma L/c^2 r^2}}{r - \gamma ma^2/c^4 r^2} = \frac{\pm 1}{\sqrt{r^3/\gamma m} \mp L/mc^2},$$

and the period of revolution of the test mass—in the form

$$T_{\pm} = 2\pi \left(\sqrt{\frac{r^3}{\gamma m}} \mp \frac{L}{mc^2} \right).$$

Let us also denote

$$T_N = \frac{T_+ + T_-}{2} = 2\pi \sqrt{\frac{r^3}{\gamma m}}, \qquad \Delta T = \frac{T_- - T_+}{2} = 2\pi \frac{L}{mc^2}.$$

The expressions given are exact, not approximate, but it should be remembered that coordinate time enters into them. It is not difficult, however, to show, using, for example, Zelmanov' s formalism (7), that the physically observable quantities will not differ essentially from the coordinate ones in a reasonable approximation. The correction to the period, ΔT, obtained and caused by the rotation of the central body, has several unexpected properties. First of all, it does not depend on the radius of the orbit and, moreover, does not contain the gravitational constant, so that its structure could have been predicted from dimensional considerations alone (the ratio of angular momentum to energy). In addition, the indicated ratio in fact does not depend on the magnitude of the mass of the central body, but only on its geometry and angular velocity. This fact, however, is not paradoxical, if one bears in mind that only the sum $T_N \pm \Delta T$ has real meaning; the first term of it clearly depends on the factors listed. Thus, for example, as the mass of the central body tends to zero, the principal part of the period T_N increases without bound, so that in the limit the addition ΔT loses all meaning. The addition to the angular velocity, on the other hand, in all cases corresponds to the Lense–Thirring frequency.

Putting the radius of the orbit of the test body $r = 3^{1/3} R$ (R is the radius of the attracting center), we find that the magnitude of the drift of the meeting point per one revolution is $\delta = 2\pi \Delta T / T_N$ and (effectively) per unit time $\delta / T = 2\pi \Delta T / T_N^2$. Thus the theory indeed predicts the planetary gravitational Zeeman effect (otherwise, the effect of partial entrainment of the "inertial frame"). We give Table 1, illustrating the magnitude of the effect for several cases.

We see that, already within the Solar System, the magnitude of the drift of the meeting point δ / T may exceed by an order of magnitude the effect of perihelion rotation

Table 1

Object (central body)	Object (central body)	Object (central body)	Object (central body)	Object (central body)	Test masses, $r = 3^{1/3}R$	Test masses, $r = 3^{1/3}R$	Test masses, $r = 3^{1/3}R$	Test masses, $r = 3^{1/3}R$	Test masses, $r = 3^{1/3}R$
name of object	m, g	R, cm	ω, sec^{-1}	L, $\frac{\text{g} \cdot \text{cm}^2}{\text{sec}}$	T_N, sec	ΔT, sec*	δ, sec. of arc**	$\frac{\delta}{T}$ **	$\frac{\Delta S_{ch}}{T}$ **
Sun	$2 \cdot 10^{33}$	$7 \cdot 10^{10}$	$3.2 \cdot 10^{-6}$	$1.3 \cdot 10^{49}$	$1.7 \cdot 10^{4}$	$4.5 \cdot 10^{-8}$	$3.3 \cdot 10^{-8}$	600	10^6
Earth	$6 \cdot 10^{27}$	$6.4 \cdot 10^{8}$	$7.2 \cdot 10^{-5}$	$7.1 \cdot 10^{40}$	$8.8 \cdot 10^{3}$	$8.3 \cdot 10^{-8}$	$1.2 \cdot 10^{-5}$	4.4	670
Jupiter	$1.9 \cdot 10^{30}$	$7.1 \cdot 10^{9}$	$1.8 \cdot 10^{-4}$	$6.7 \cdot 10^{45}$	$1.8 \cdot 10^{4}$	$2.5 \cdot 10^{-5}$	$2.5 \cdot 10^{-4}$	280	10^4
Rapidly rotating star	$3 \cdot 10^{34}$	$3.5 \cdot 10^{11}$	$1.5 \cdot 10^{-4}$	$2 \cdot 10^{53}$	$5 \cdot 10^{4}$	$4.7 \cdot 10^{-2}$	1.21	$7.6 \cdot 10^{4}$	10^6
Neutron star	10^{33}	10^{6}	10^{4}	$4 \cdot 10^{48}$	$1.3 \cdot 10^{-3}$	$2.8 \cdot 10^{-5}$	$2.5 \cdot 10^{4}$	93 $\frac{\text{rad}}{\text{sec}}$	750 $\frac{\text{rad}}{\text{sec}}$

* The same for all r.
** In seconds of arc per century.

For Mercury, although, of course, such an effect of perihelion rotation for the orbits considered (if their eccentricity were different from zero) in the corresponding cases exceeds the magnitude of the drift effect (see the last column of the table). However, both effects—the one due to the rotation of the central body and the one associated with the Schwarzschild part of its gravitational field —approach each other in order of magnitude for rapidly rotating stars (stars of classes A, O, and especially B; the example of an extremely rapidly rotating neutron star is taken from work (3), and it should be noted that the known pulsars have angular velocities of rotation smaller by two or more orders of magnitude).

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Note: Figure translations are in progress. See original paper for figures.