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# MATHEMATICS

Academician G. I. MARCHUK, S. A. ATANBAEV

1970

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**Abstract**

**Full Text**

## MATHEMATICS

Academician G. I. MARCHUK, S. A. ATANBAEV

### SOME QUESTIONS OF “GLOBAL”REGULARIZATION

1°. Let  $G$  be an  $m$ -dimensional domain with boundary  $\Gamma_0$ . Denote by  $L_2(G)$  the space of all functions square-summable over the domain  $G$  with norm

$$\|u\|^2 = \int \dots \int |u|^2 dx, \quad u = u(x, t), \quad x = (x_1, x_2, \dots, x_m),$$

$0 \leq t \leq T$ . In the domain  $\Omega = G \times [0, T]$  with lateral surface  $\Gamma = \Gamma_0 \times [0, T]$ , consider the problem

$$\partial u / \partial t = \mathcal{L}u, \tag{1}$$

$$u(x, 0) = u_0(x), \quad x \in G, \tag{2}$$

where  $\mathcal{L}$  is a linear positive definite differential operator of order  $2(s-1)$ , which is self-adjoint under the boundary conditions

$$\mathcal{L}_1(u) = \mathcal{L}_2(u) = \dots = \mathcal{L}_{s-1}(u) = 0. \tag{3}$$

Here  $\mathcal{L}_i$  are certain differential operators. The posed problem is well-posed in the sense of Tikhonov in the class of solutions uniformly bounded in  $t$  in the metric  $L_2(G)$  <sup>(1)</sup>.

For an approximate solution of problem (1)–(3), in papers <sup>(2,3)</sup> a local method of computation is used, according to which equation (1) is replaced by the difference equation

$$(u^k - u^{k-1}) / \Delta t = \mathcal{L}u^{k-1}, \tag{4}$$

where  $\Delta t = T/M$ ,  $u^k = u(x, k\Delta t)$  ( $k = 1, 2, \dots, M$ ), and equation (4) is solved for each fixed  $k$ .

In the present note an attempt is made to apply a global method of computation, whose idea consists in the following. Suppose that a difference grid with corresponding uniform steps  $h_i$  ( $i = 1, 2, \dots, m$ ) has been introduced in the domain  $G$ . We replace equation (1) by the difference equation

$$(u^k - u^{k-1}) / \Delta t = \beta \mathcal{L}_h u^k + (1 - \beta) \mathcal{L}_h u^{k-1} \quad (0 < \beta \leq 1).$$

Next, we write out the difference equation at all interior grid nodes for all time instants from the interval  $0 \leq t \leq T$ . Taking into account the boundary conditions (3), the problem reduces to solving a system of linear algebraic equations

$$A\varphi = f, \tag{5}$$

where  $A$  is a square matrix, and  $\varphi$  and  $f$  are grid vector-functions. We shall assume that the solution of equation (5) exists. Using Lanczos' method, we symmetrize equation (5). Then we obtain

$$\Lambda\varphi = \mathbf{f}, \quad (6)$$

where

$$\Lambda = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \quad \varphi = \{\varphi^*, \varphi\}', \quad \mathbf{f} = \{f, f^*\}',$$

$\mathbf{f}$  is a certain vector-function. It is known that if  $\{\mu_q\}$  are the eigenvalues of the matrix  $\Lambda$ , then

$$\max_q |\mu_q| = [\rho(A^*A)]^{1/2}.$$

Let  $D^0$  be a closed set of vectors  $\{\varphi^j\}$ , and  $D_\Lambda^0 = \{\xi^j : \xi^j = \Lambda\varphi^j - \mathbf{f}, \varphi^j \in D^0\}$  a linear space. To solve the equation

(6) we formulate a convergent iterative process

$$\varphi^{j+1} = \varphi^j - \sum_{i=1}^p \gamma_{ij} \Lambda^{i-1} \xi^j, \quad \varphi^j \in D^0 \quad (j = 0, 1, 2, \dots), \quad p \geq 2, \quad (7)$$

where  $\xi^j = \Lambda\varphi^j - \mathbf{f}$ , and  $\gamma_{ij} = \gamma_{ij}(\xi^j)$  ( $i = 1, 2, \dots, p$ ) are real numbers that are the solution of a certain compatible system (4). By convergence of the iterative process (7) in  $D^0$  we shall understand the convergence of the sequence of vectors  $\{\varphi^j\}$  to the vector  $\varphi^* = \Lambda^{-1}\mathbf{f}$ . The functional  $\Phi_j = (\Lambda\varphi^j, \mathbf{f})/(\Lambda\varphi, \mathbf{f}) = (\Lambda^2\varphi^j, \varphi)/(\Lambda^2\varphi, \varphi)$  characterizes, in a certain sense, the closeness of the approximate solution to the exact one. Transform  $\Phi_j$  to the form  $\Phi_j = 1 + (\xi^j, \mathbf{f})/(\mathbf{f}, \mathbf{f})$ . Consequently, the condition  $\|\xi^j\| \leq \delta\|\mathbf{f}\|$ , where  $\delta$  is an a priori constant tending to zero, is a necessary condition, and, if  $\Lambda$  is a positive definite matrix, also a sufficient condition for convergence of the iterative process.

2°. The convergence of the approximate solution of equation (6) to the solution of problem (1)–(3) is slow. To accelerate the convergence of the iterative process one may use regularization methods (5–7). In the present case we shall follow (7).

Instead of problem (1)–(3), consider the problem

$$\partial v / \partial t = \mathcal{L}_{(\alpha)} v, \quad (8)$$

$$v(x, 0) = v_0(x), \quad \mathcal{L}_1(v) = \mathcal{L}_2(v) = \dots = \mathcal{L}_{2(s-1)}(v) = 0, \quad (9)$$

where  $\mathcal{L}_{(\alpha)} = \mathcal{L} - \alpha\mathcal{L}^2$ ,  $\alpha$  is a positive parameter ( $\alpha \rightarrow 0$ ). Problem (8)–(9) has a unique solution, which converges to  $u(x, t)$  as  $\alpha \rightarrow 0$  (7). Following the idea

of the global method of computation, we reduce problem (8)–(9) to a system of equations. We carry out the symmetrization of the operator analogously to (6), and as a result obtain the system of algebraic equations

$$\Lambda_\alpha \varphi_\alpha = \mathbf{f}_\alpha, \quad (10)$$

for the solution of which we use the iterative process (7).

We shall now dwell on one iterative method for choosing the parameter  $\alpha$ . Let  $\Lambda_\alpha = \Lambda + \alpha P$ , where  $P$  is a positive definite matrix corresponding to the difference approximation of  $\mathcal{L}^2$ . Consider the iterative process

$$\varphi_{\alpha_{j+1}}^{j+1} = \varphi_{\alpha_j}^j - \tau_j (\Lambda \varphi_{\alpha_j}^j - \mathbf{f}_{\alpha_j} + \alpha_j P \varphi_{\alpha_j}^j), \quad \varphi_{\alpha_j}^j \in D^0 \quad (j = 0, 1, \dots). \quad (11)$$

The iterative process (11) may be regarded as a process depending on two parameters  $\alpha_j$  and  $\tau_j$ , which can be chosen at each step so as to minimize the norm of the residual  $\xi^j = \Lambda \varphi_{\alpha_j}^j - \mathbf{f}_{\alpha_j}$ . Such minimization leads (8) to the choice of the parameters  $\alpha_j$  and  $\tau_j$  in the form

$$\alpha_j = \frac{(\Lambda \xi^j, \xi^j)(\Lambda \xi^j, \Lambda \psi^j) - (\Lambda \psi^j, \xi^j) \|\Lambda \xi^j\|^2}{(\Lambda \psi^j, \xi^j)(\Lambda \xi^j, \Lambda \psi^j) - (\Lambda \xi^j, \xi^j) \|\Lambda \psi^j\|^2}, \quad \tau_j = \frac{(\Lambda \xi^j, \xi^j) + \alpha_j (\Lambda \psi^j, \xi^j)}{\|\Lambda \xi^j + \alpha_j \Lambda \psi^j\|^2},$$

where  $\psi^j = P \varphi_{\alpha_j}^j$ . Obviously,  $\alpha_j \rightarrow 0$  as  $\|\xi^j\| \rightarrow 0$ . This ensures the convergence of the iterative process to the solution of the original problem.

3°. As an example, consider the Cauchy problem for the heat-conduction equation with reverse time flow. Let  $\bar{f}(x, t)$  be a function with domain of definition  $\Pi : \{0 \leq x \leq 1, 0 \leq t \leq T\}$ , belonging to some Hilbert space  $G_2[0, 1]$  with norm

$$\|f(x, t)\|_{G_2}^2 = \int_0^1 \bar{f}^2(x, t) dx \quad (0 \leq t \leq T).$$

Denote by  $L_2(0, T; G_2)$  the space of functions  $f(x, t)$  square-summable on  $[0, T]$ , i.e.  $f(x, t) \in L_2(0, T; G_2)$ , if

$$\int_0^T \|f(x, t)\|_{G_2}^2 dt = \|f\|_{L_2(0, T; G_2)}^2 < \infty.$$

Let  $\bar{G}_2 \subset G_2$  and let  $\bar{G}_2$  be dense in  $G_2$ .

In the rectangle  $\Pi$  consider the following two problems:

$$u_t = -u_{xx} + \bar{f}(x, t); \quad (12)$$

$$u(x, 0) = \eta(x), \quad u(0, t) = u(1, t) = 0; \quad (13)$$

$$v_t = -v_{xx} - \alpha v_{xxxx} + \bar{f}(x, t); \quad (14)$$

$$v(x, 0) = \eta(x), \quad v(0, t) = v_{xx}(0, t) = v(1, t) = v_{xx}(1, t) = 0. \quad (15)$$

If  $\bar{f}(x, t) \in L_2(0, T; \overline{G}_2)$  and  $\alpha > 0$ , then the (7) problem (14)–(15) has a unique solution  $v(x, t)$  with the properties

$$v(x, t) \in L_2(0, T; \overline{G}_2^*), \quad v_t(x, t) \in L_2(0, T; \overline{G}_2^*),$$

and it converges to the solution of problem (12)–(13) as  $\alpha \rightarrow 0$ .

Construct in the rectangle  $\Pi$  a uniform mesh  $\omega_{h\tau} = \{(x_i = ih, t_n = n\tau), i = 0, 1, \dots, N; n = 0, 1, \dots, M\}$  with steps  $h = 1/N$  and  $\tau = T/M$ . For the approximate solution of problem (14)–(15) consider the difference scheme

$$\begin{aligned} v_i^{n+1} - v_i^n &= -\frac{\tau}{h^2} (v_{i-1}^{n+1} - 2v_i^{n+1} + v_{i+1}^{n+1}) \\ &- \frac{\tau\alpha}{h^4} (v_{i-2}^{n+1} - 4v_{i-1}^{n+1} + 6v_i^{n+1} - 4v_{i+1}^{n+1} + v_{i+2}^{n+1}) + \tau \bar{f}_i^{n+1}; \end{aligned} \quad (16)$$

$$v_i^n = v(ih, n\tau), \quad \bar{f}_i^n = \bar{f}(ih, n\tau), \quad v_i^0 = \eta(ih), \quad (17)$$

$$v_0^n = v_{-1}^n + v_1^n = v_{N-1}^n + v_{N+1}^n = v_N^n = 0.$$

Writing out, according to the method of global counting, the difference equation (16) on the mesh  $\bar{\omega}_{h\tau} = \{x_i = ih, t_n = n\tau\}, i = 1, 2, \dots, N-1; n = 0, 1, \dots, M-1$  and taking into account the difference initial and boundary conditions (17), we obtain the system of equations

$$AV = \tau F \quad (18)$$

with block square matrix  $A = \{A_{kl}\}$  of order  $M$  with elements  $A_{kk} = B$ ,  $A_{kl} = -E$  for  $k - l = 1$ , and  $A_{kl} = 0$  for  $l > k$  and  $k - l \geq 2$ , where  $E$  is the identity matrix of order  $N - 1$ , and  $B = \{b_{ij}\}$  is a symmetric square matrix of the same order with elements

$$b_{ij} = \left\{ \frac{\alpha\tau}{h^4} (\delta_{i-2}^j + \delta_{i+2}^j) + \frac{\tau}{h^2} \left( 1 - \frac{4\alpha}{h^2} \right) (\delta_{i-1}^j + \delta_{i+1}^j) + \left( 1 + \frac{6\alpha\tau}{h^4} - 2\frac{\tau}{h^2} \right) \delta_i^j \right\} \quad (ij \neq 1, \quad i \times j \neq (N-1)^2),$$

$$b_{11} = b_{N-1, N-1} = 1 + 5\alpha\tau/h^4 - 2\tau/h^2,$$

where  $\delta_{i_0}^{j_0} = 1$  for  $|i_0 - j_0| \leq 2$  and  $\delta_{i_0}^{j_0} = 0$  for  $|i_0 - j_0| > 2$ . In equation (18),  $V = \{V^1, V^2, \dots, V^M\}$ ,  $F = \{F^1, F^2, \dots, F^M\}$  are mesh vector-functions with components  $V^n = \{v_1^n, v_2^n, \dots, v_{N-1}^n\}$  ( $n = 1, 2, \dots, M$ ),

$$F^n = \{\bar{f}_1^n, \bar{f}_2^n, \dots, \bar{f}_{N-1}^n\} \quad (n = 2, 3, \dots, M), \quad F^1 = \bar{f}^1 + \frac{1}{\tau}V^0,$$

$$\bar{f}^1 = \{\bar{f}_1^1, \bar{f}_2^1, \dots, \bar{f}_{N-1}^1\}.$$

Let  $\mathcal{H}_{h\tau}$  be the real vector Euclidean space of mesh vector-functions with elements  $\{V^n\}$  and  $\{F^n\}$ , where the scalar product and the norm are defined in the form

$$(V^{n_1}, V^{n_2}) = \sum_{i=1}^{N-1} v_i^{n_1} \times v_i^{n_2}, \quad \|V^n\| = [(V^n, V^n)]^{1/2}.$$

**Lemma.** If  $h$ ,  $\tau$ ,  $a$  are chosen so that  $h^2 \leq 8a$  and  $\tau < 4a$ , then

1) all eigenvalues

$$\lambda_k = 1 - \frac{4\tau}{h^2} \sin^2 \frac{k\pi h}{2} \left( 1 - \frac{4a}{h^2} \sin^2 \frac{k\pi h}{2} \right) \quad (k = 1, 2, \dots, N-1)$$

of the matrix  $B$  are positive, and

$$c = \min_k \lambda_k = \lambda_{k_0} = 1 - \frac{\tau}{4a} \quad \text{for} \quad k_0 = \frac{2}{\pi h} \arcsin \frac{h}{2\sqrt{2a}};$$

2) the matrices  $B$  and  $\Lambda$  are nonsingular.

The validity of assertion 1) is established by a direct check of the extremum of the function  $\lambda_k = g(k)$  as a function of the continuous argument  $k$ , while the validity of 2) follows from Hadamard' s regularity condition (9) and from the equality  $\det \Lambda = (\det B)^M$ . It can be established that the difference equation (16) approximates the differential equation (14) with accuracy  $O(\tau + h^2)$ .

**Theorem 1.** If  $\tilde{f}_0 = \sup \|F^n\|$ , then, under the conditions of the lemma, the difference boundary-value problem (16)–(17) is stable in  $H_{h\tau}$ , and the estimate

$$\|V^M\| \leq \exp\left(\frac{T}{4ca}\right) \|V^0\| + 4a\tilde{f}_0 \left(\exp\left(\frac{T}{4ca}\right) - 1\right).$$

In order to symmetrize equation (18), introduce the vector-functions  $\mathbf{V} = \{V^*, V\}'$ ,  $\mathbf{F} = \{F, F^*\}$ , where  $V^* = \{V^M, V^{M-1}, \dots, V^1\}$  and  $F^* = \{F^M, F^{M-1}, \dots, F^1\}$ . Then we obtain

$$\Lambda \mathbf{V} = \tau \mathbf{F}. \quad (19)$$

**Theorem 2.** The multistep iterative process (7) for equation (19) converges to its exact solution, and the estimate

$$\|\mathbf{V} - \mathbf{V}^j\| \leq \|\xi^j\|/\mu_0,$$

holds, where  $\mu_0$  is the eigenvalue of the matrix  $\Lambda$  smallest in absolute value.

**Remark.** As  $h$  and  $\tau \rightarrow 0$ , the order of the matrix  $\Lambda$  increases. Therefore, in the global computational method there arises the question of economizing computer memory in connection with the storage of the matrix  $\Lambda$ . For the practical realization of algorithms (7) and (11), it will be necessary to compute the vector  $\Lambda \mathbf{V} = \mathbf{W}$ , where  $W = \{W^*, W\}$ ,  $W = \{W^1, W^2, \dots, W^M\}$ ,  $W^* = \{W^M, W^{M-1}, \dots, W^1\}$ .

Since  $\Lambda$  and  $\mathbf{V}$  are, respectively, a block matrix and a vector-function, the computation of the vector  $\mathbf{W}$  reduces to the blockwise computation of its components with the aid of the symmetric 5-diagonal matrix  $B$ , according to the formula

$$W^1 = BV^1, \quad W^n = BV^n + [\gamma E - (\gamma - 1)B]V^{n-1} \quad (n = 2, 3, \dots, M).$$

The numerical experiments carried out for problem (14)–(15) showed the effectiveness of the proposed method.

An essential circumstance of the global computational method is its very weak dependence on the choice of  $\alpha$ , whereas in the local computational method the results, as a rule, depend substantially on the choice of  $\alpha$ .

Computing Center  
Siberian Branch of the Academy of Sciences of the USSR  
Novosibirsk

Received  
7 VII 1969

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