TWO EXAMPLES IN THE THEORY OF DIMENSION OF BICOMPACTA

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TWO EXAMPLES IN THE THEORY OF DIMENSION OF BICOMPACTA

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In the present note two bicompacta are constructed. In § 1 a completely normal bicompactum Z is constructed for which $\dim Z = 1$, $\operatorname{ind} Z = \operatorname{Ind} Z = 2$. In § 2 a bicompactum Φ with the first axiom of countability is constructed for which $\dim \Phi = 1$, $\operatorname{ind} \Phi = \operatorname{Ind} \Phi = 2$, and moreover the small inductive dimension is equal to 2 at every point of the bicompactum Φ.

In constructing the bicompactum Z one has to use Suslin' s hypothesis on the existence of an ordered, completely normal, nonseparable connected bicompactum, as well as the continuum hypothesis, which asserts that $2^{\aleph_0} = \aleph_1$. In Cohen' s paper ([2]) it is proved that the continuum hypothesis may be adopted as an axiom of set theory. Tennenbaum ([3]) proved the same thing with respect to Suslin' s hypothesis.

§ 1. 1. In what follows it is more convenient for us to deal with an ordered, connected, completely normal nonseparable bicompactum S, no open subset of which is separable. From the existence of an arbitrary Suslin continuum there follows the existence of the required one.

2. The product $Z^* = S \times I$ of the Suslin continuum with the ordinary interval, as is easily seen, is a completely normal bicompactum, and therefore its weight and the cardinality of the set of its closed subsets do not exceed $2^{\aleph_0}$ ([1]).
3. In Z^* one can choose a subset everywhere dense in Z^* such that its intersection with any nowhere dense closed subset is at most countable.

This can be done as follows. Number by transfinite ordinals smaller than ω_1 the set of all nowhere dense closed subsets $\{F_\alpha\}$, $\alpha < \omega_1$, of the bicompactum Z^*, and some base $\{U_\alpha\}$, $\alpha < \omega_1$, of cardinality $\aleph_1 = 2^{\aleph_0}$ of the bicompactum Z. For each $\alpha < \omega_1$ take such a point z^*_α that $z^*_\alpha \notin \bigcup_{\beta \leq \alpha} F_\beta$ and $z^*_\alpha \in U_\alpha$. Such a point z^*_α can be chosen, since in the bicompactum Z^* no open set is representable as the union of nowhere dense sets. We may assume that $z^*_\alpha \notin S \times 1$ and $z^*_\alpha \in S \times 0$ for every $\alpha < \omega_1$. The set $M = \{z^*_\alpha, \alpha < \omega_1\}$ of the selected points has the required property (this is nothing other than Lusin' s ν-set).

4. For every point z^* of the bicompactum Z^* there exists a monotone mapping f_{z^*} of the bicompactum Z^* onto the square I^2 which is one-to-one at the point z^*, i.e., the set $f_{z^*}^{-1} f_{z^*}(z^*)$ consists of one point, and such that the preimage of a base at the point $f_{z^*}(z^*)$ will be a base at the point $z^* \in Z^*$.

5. Let the point z^* belong to the set M. Consider the mapping f_{z^*} of the bicompactum Z^* onto the square I^2. We insert in Z^* and I^2, instead of the points z^* and $f_{z^*}(Z^*)$, circles. For the points of the circle inserted in the square I^2, neighborhoods are defined in the same way as this is done in ([4]). Let $f_{z^*}(z^*) = y$. Points of the circle inserted instead of the point y will be denoted by (y, φ). As a neighborhood of the point (y, φ) we take an arc of the circle

$((y, \varphi_1), (y, \varphi_2))$, containing our point (y, φ) and a part of a circular sector of some radius on the square, enclosed between the rays issuing from the point y at the angles φ_1 and φ_2. For all the remaining points of the square the neighborhoods are the previous ones. For the points of the circle (z^*, φ) inserted into the bicompactum Z^*, we define neighborhoods as the inverse images of the neighborhoods of the corresponding points on the square. It is easily verified that the space thus obtained is again a bicompactum.

Now, for each point of the set M, instead of it, in an analogous way we insert a circle and likewise define the neighborhoods of points on these circles. Moreover, if, in defining a neighborhood for some point, a point from the set M falls into the neighborhood, then we assume that the whole circle by which this point is replaced also belongs to the neighborhood being defined. The newly obtained space Z is a perfectly normal bicompactum. The proof of the last assertion is analogous to that given in the work [4].

6. The natural mapping π of the bicompactum Z onto the bicompactum Z^*, which glues the inserted circles back into points, is monotone and irreducible.

By a horizontal (vertical) in the bicompactum Z we shall mean any bicompact set B whose image under the mapping π is a horizontal (vertical) in the bicompactum Z^*. Moreover, if $\pi(B)$ contains a point z^* of the set M, then the set B must contain only the points $(z^*, 0)$ and (z^*, π) ($(z^*, {}^1/_2\pi)$ and $(z^*, {}^3/_2\pi)$) from the corresponding circle, and we shall say that the horizontal (vertical) B passes through the indicated circle.

7. We shall show that $\operatorname{Ind} Z = 2$. It is clear that $\operatorname{Ind} Z \leqslant 2$, and it remains to prove that $\operatorname{Ind} Z \geqslant 2$. Let A be any nowhere dense partition between the sets $S \times 1$ and $S \times 0$ in the bicompactum Z. The set $\pi(A)$ is a nowhere dense partition between the sets $S \times 1$ and $S \times 0$ in the bicompactum Z^*; therefore it contains no more than a countable number of points $z_1^*, z_2^*, \dots, z_n^*, \dots$ of the set M. At the remaining points of the set $\pi(A)$ the mapping π is a homeomorphism. Let $s_1, s_2, \dots, s_n, \dots$ be the projections of the points $z_1^*, z_2^*, \dots, z_n^*, \dots$ on the bicompactum S. Since the continuum S is not

separable, there is in it an interval $(\tilde{s}_1, \tilde{s}_2)$ containing none of the points $s_1, s_2, \dots, \dots, s_n, \dots$. Consider the bicompactum $[\tilde{s}_1, \tilde{s}_2] \times I = Z_1^*$. The set $\pi(A) \cap Z_1^*$ is a partition between the sets $[\tilde{s}_1, \tilde{s}_2] \times 1$ and $[\tilde{s}_1, \tilde{s}_2] \times 0$ in the bicompactum Z_1^*. By standard arguments one can show that the set $\pi(A) \cap Z_1^*$ contains a one-dimensional bicompactum, and since the mapping π is a homeomorphism on this set, a one-dimensional bicompactum is also contained in the set A. Thus, $\operatorname{Ind} Z \geqslant 2$.

Since every open subset of the continuum S is nonseparable, analogous reasoning shows that at each point z of the bicompactum Z the small inductive dimension is equal to 2, i.e. $\operatorname{ind}_z Z = 2$.

8. We shall prove that $\dim Z = 1$. Let $\omega = \{U_1, U_2, \dots, U_p\}$ be an arbitrary finite open covering of the bicompactum Z. The number of circles $\{K_1, K_2, \dots, K_r\}$, each of which does not belong wholly to any element of the covering, is finite; otherwise, taking on each circle a point and choosing a convergent subsequence $z_1^*, z_2^*, \dots, \dots, z_n^*, \dots$, we would obtain that the element of the covering ω containing the limiting point contains almost all the indicated circles. We inscribe into the original covering ω an open covering $\gamma = \{V_1, V_2, \dots, V_m\}$ such that the stars of the circles $K_1, K_2, \dots, K_r$ do not intersect. We inscribe into the covering the circle K_i $(i = 1, 2, \dots, r)$, cut out by the covering γ, a finite refinement of multiplicity 2 and extend it to a refinement of multiplicity 2 of some neighborhood O_i of this circle. Take a rectangle Π_i $(i = 1, 2, \dots, r)$ with sides parallel to the factors, which contains the circle K_i and is contained in its neighborhood O_i. We shall now draw horizontals and verticals, including also the extensions of the sides of the recta-

rectangles Π_i $(i = 1, 2, \dots, r)$, so that the partition determined by them is inscribed in the covering γ; moreover, we discard those parts of these vertical and horizontal lines that intersect the rectangles Π_i $(i = 1, 2, \dots, r)$, and on the rectangles Π_i $(i = 1, 2, \dots, r)$ we take the partition previously constructed there. As a result we obtain a partition $\nu = \{D_1, D_2, \dots, D_l\}$, which in general has multiplicity 4, and on the rectangles Π_i $(i = 1, 2, \dots, r)$ has multiplicity 2. Let the points at which the partition has multiplicity 3 or 4 be $z_1, z_2, \dots, z_{n_1}$. Suppose that the point z_1 belongs to four elements of the partition $D_{i_1}, D_{i_2}, D_{i_3}$ and D_{i_4}. The boundaries of these elements of the partition form a "cross" at the point z_1, i.e. a certain piece of a vertical and a horizontal line.

Take a rectangle $\widetilde{\Pi}_1$ containing only the point z_1 and belonging to the intersection of elements of the covering γ which contain the elements of the partition $D_{i_1}, D_{i_2}, D_{i_3}$ and D_{i_4}. In the lower right corner of the rectangle $\widetilde{\Pi}_1$ choose a circle and draw through it a vertical and a horizontal line up to their intersection with the boundary of the rectangle $\widetilde{\Pi}_1$. Then $\widetilde{\Pi}_1$ is divided into 4 rectangles: $\widetilde{\Pi}_1^1$—the upper left, $\widetilde{\Pi}_1^2$—the upper right, $\widetilde{\Pi}_1^4$—the lower left, $\widetilde{\Pi}_1^3$—the lower right. To the upper left element of the partition D_{i_1} we add the rectangle $\widetilde{\Pi}_1^1$, i.e.

$$\widetilde{D}_{i_1} = \left(D_{i_1} \setminus \bigcup_{i=2}^{4} \widetilde{\Pi}_1^i \right) \cup \widetilde{\Pi}_1^1.$$

Instead of the upper right element D_{i_2} of the partition ν we take

$$\widetilde{D}_{i_2} = \left(D_{i_2} \setminus \bigcup_{\substack{i=1 \\ i \neq 2}}^{4} \widetilde{\Pi}_1^i \right) \cup \widetilde{\Pi}_1^2,$$

instead of the lower left element D_{i_4} we take

$$\widetilde{D}_{i_4} = \left(D_{i_4} \setminus \bigcup_{i=1}^{3} \widetilde{\Pi}_1^i \right) \cup \widetilde{\Pi}_1^4,$$

and instead of the lower right element D_{i_3} we take the set

$$\widetilde{D}_{i_3} = \left(D_{i_3} \setminus \bigcup_{\substack{i=1 \\ i \neq 3}}^{4} \widetilde{\Pi}_1^i \right) \cup \widetilde{\Pi}_1^3.$$

Now, instead of the elements of the partition $D_{i_1}, D_{i_2}, D_{i_3}$ and D_{i_4}, we have the elements of the partition $\widetilde{D}_{i_1}, \widetilde{D}_{i_2}, \widetilde{D}_{i_3}$ and $\widetilde{D}_{i_4}$ of multiplicity 2, and no new points at which the multiplicity is ≥ 3 have appeared. Thus, carrying out a finite number of rearrangements of the partition ν at the points $z_1, z_2, \dots, z_{n_1}$, we obtain a partition of multiplicity 2 inscribed in the covering γ, and consequently also in the initial covering ω. This proves our assertion.

§ 2. 1. Consider the set of countable sequences $\{a_1, a_2, \dots\}$ of numbers of the interval $[0, 1]$, and take only those sequences in which, if a 0 or a 1 occurs at some place, then the same number stands in all subsequent places. The topology on the set of all such sequences is introduced by lexicographic ordering of them: $\{a_i,\ i = 1, 2, \dots\} < \{\beta_i,\ i = 1, 2, \dots\}$ if the first number at which these sequences differ is smaller in the first than in the second. After introducing the topology we obtain an ordered continuum L with the first axiom of countability. The set B of sequences in which 0 or 1 occurs is everywhere dense in L. Every interval of the continuum L contains a continuum homeomorphic to it. Therefore the continuum L can be represented as follows: in the interval $[0, 1]$ each point, except for the endpoints, is replaced by the continuum L.

2. Let the bicompactum Φ^* be the product of the continuum and the interval $I = [0, 1]$. The interval $(0, 1)$ can be represented as the sum of a continuum number of everywhere dense mutually disjoint sets P^ξ, $\xi \in \Omega$. Renumber all points $\langle l^\xi \rangle$, $\xi \in \Omega$, of the continuum L in whose notation as sequences of

numbers neither 0 nor 1 occurs. Take on the segment $l^\xi \times I$ the set $l^\xi \times P^\xi$. Denote the sum of all such sets by D. This set is everywhere dense in the bicompactum Φ^*. Note that any horizontal line in the bicompactum Φ^* contains only one point of the set D, with the exception of the horizontal lines $L \times 0$ and $L \times 1$.

3. Just as in the preceding paragraph, in the bicompactum Z^* replace each point d of the set D by a circle and define neighborhoods of points

of these neighborhoods by means of mutually one-to-one mappings of the bicompactum Φ^* onto the square I^2 at the corresponding point d. We obtain a bicompactum Φ. The natural mapping π of the bicompactum Φ onto the bicompactum Φ^* is continuous, monotone, and irreducible.

4. At every point y of the bicompactum Φ we have

$$\operatorname{ind}_y \Phi = 2.$$

Indeed, let Oy be an arbitrary neighborhood of the point y. We may assume that this neighborhood does not meet the lower or upper base of the bicompactum Φ. We shall suppose that the open set Oy does not meet the upper base. Take in the set Oy a point $y_1 = (l_1, \alpha_1)$ of the bicompactum Φ, whose coordinate l_1 belongs to the set B of the continuum L. The horizontal $L \times \alpha_1$ meets only one circle, and therefore one can take on it a point $y_2 = (l_2, \alpha_1)$ so that the part of the horizontal $[(l_2, \alpha_1), (l_1, \alpha_1)]$ does not meet the circle and belongs to the neighborhood Oy, and the point y_2 may be chosen so that the coordinate l_2 also belongs to the set B. Consider the rectangle

$$\Pi = [l_2, l_1] \times [\alpha_1, 1]$$

in the bicompactum Φ^* and its inverse image $\pi^{-1}(\Pi) = P$ in the bicompactum Φ. The set $\operatorname{Fr}(Oy) \cap P = A$ is a nowhere dense partition between the upper and lower bases in the bicompactum P.

We shall show that $\operatorname{ind} A \geq 1$. The set $\pi(A)$ is a nowhere dense partition between the upper and lower bases in the rectangle Π. We shall suppose that $\pi(A)$ is connected. Let

$$\Pi \setminus \pi(A) = V_1 \cup V_2,$$

where V_1 and V_2 are open, $V_1 \cap V_2 = \emptyset$, and V_1 contains the top, while V_2 contains the bottom of the rectangle Π. The mapping π is a homeomorphism on the set of points of one-to-one correspondence, and it is non-one-to-one only at the points $\{d\}$ of the set D of the bicompactum Φ^*. Consequently, it is enough to show that the set $\pi(A)$ contains a connected bicompactum not containing points of the set D. If the set $\pi(A)$ contains a horizontal segment, then in it there is a smaller segment not containing points of the set D. Suppose now that the set $\pi(A)$ contains no horizontal segment. Represent the continuum $[l_2, l_1]$ of the bicompactum Φ^* in the same way as the continuum L: in the segment $[0, 1]$ each point t is replaced by a continuum $\tilde{L}$ homeomorphic to L. As the points

t of the segment $[0,1]$ take points from the set B. The continuum $\widetilde{L}$, inserted in place of the point t, will be denoted by $\widetilde{L}_t$. Denote the projections of the bicompactum Π onto its factors by π_1 and π_2, respectively. For each rectangle $\widetilde{L}_t \times [0,1]$ put

$$\alpha_1(t) = \sup\{\pi_2(y),\ y \in (\widetilde{L}_t \times [0,1]) \cap \pi(A)\},$$

$$\alpha_2(t) = \min\{\pi_2(y),\ y \in (\widetilde{L}_t \times [0,1]) \cap \pi(A)\},$$

with $\alpha_1(t) > \alpha_2(t)$. Each segment

$$\widetilde{L}_t \times \alpha \qquad (\alpha_1(t) < \alpha < \alpha_2(t))$$

meets the sets V_1 and V_2. Taking for each t the segment $[\alpha_1(t), \alpha_2(t)]$, we obtain an uncountable set of segments. Since the segment $[0,1]$ has a countable base, there is a segment $[h_1, h_2]$ contained in an infinite set of segments $[\alpha_1(t_i), \alpha_2(t_i)]$, $i = 1, 2, \ldots$. Let the sequence $t_1, t_2, \ldots$ converge to the point $\hat{t}$ (otherwise we choose a convergent subsequence). Then the segment

$$[(\hat{t}, h_1), (\hat{t}, h_2)]$$

belongs to the set $\pi(A)$, since every neighborhood of this segment contains points of the sets V_1 and V_2. Since the segments $t \times [0,1]$ do not contain points of the set D, this proves that $\operatorname{Ind} A \geq 1$. Consequently, it has been proved that $\operatorname{ind} \Phi \geq 2$. It is evident that

$$\operatorname{ind} \Phi \leq \operatorname{Ind} \Phi \leq 2.$$

Thus,

$$\operatorname{ind} \Phi = \operatorname{Ind} \Phi = 2.$$

5. The fact that

$$\dim \Phi = 1$$

is proved in exactly the same way as for the bicompactum Z in § 2.

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Note: Figure translations are in progress. See original paper for figures.