ON A FORMULA FOR REPRESENTING THE FUNDAMENTAL SOLUTION OF A DIFFERENTIAL EQUATION BY A CONTINUAL INTEGRAL

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Abstract

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ON A FORMULA FOR REPRESENTING THE FUNDAMENTAL SOLUTION OF A DIFFERENTIAL EQUATION BY A CONTINUAL INTEGRAL

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It has long been known and widely used that a formula represents the fundamental solution of the equation

$$\frac{1}{h}\frac{\partial u}{\partial t} = \Delta u + q(x)u \qquad (\operatorname{Re} h > 0) \tag{1}$$

in the form of a continual integral (see, for example, the survey article ([1])). In view of the broad possibilities for applying this formula, attempts have repeatedly been made to generalize it to other classes of equations. In particular, in the same article ([1]) the wish was expressed to define a certain measure by means of the equation $\partial u/\partial t + \partial^4 u/\partial x^4 = 0$ and to represent the solution of the equation $\partial u/\partial t + \partial^4 u/\partial x^4 + q(x)u = 0$ as an integral of some functional with respect to this measure. Soon after this, Yu. L. Daletskii ([2–4]) constructed the required measures for a very broad class of operator equations of the form $\partial u/\partial t = Au$ and obtained a number of formulas with integrals with respect to these measures; somewhat later V. Yu. Krylov ([5]), unaware of Daletskii' s work, proposed a solution to the problem posed in ([1]). However, although Daletskii' s works answered the question posed very fully, the formulas he obtained did not find particularly interesting applications. It seems to me that the very approach to the problem proposed in article ([1]) was unsuccessful. Indeed, the most fruitful direction in using continual integrals for equation (1) is connected with the study of their asymptotic behavior for small values of the parameter h. This study is carried out, roughly speaking, by the method of descent, and for the successful application of this method it is desirable that under the integral there stand an exponential with a comparatively simple exponent; moreover, this condition applies equally to the measure and to the functional integrated with respect to it. In the classical cases of integrals with respect to Wiener measure or Feynman measure, the above condition is satisfied, whereas in Daletskii' s formulas it is not. In general, if one adopts this point of view, there is no particular sense in singling out any part of the equation, constructing a measure with its help, and studying integrals of functionals with respect to this measure.

In the present work a continual integral is written down which gives the fundamental solution of a differential equation for a fairly broad class of equations; moreover, under the integral in all cases there stands an exponential with a comparatively simple exponent. The formula obtained differs from the known formulas* even for the case of equation (1).

* Only after submitting the note to *Doklady Akademii Nauk* did I learn from E. E. Shnol that the formula mentioned was obtained by Feynman ([7]) as early as 1951 and was subsequently rediscovered several more times. All these rediscoveries (as well as the original result) were made on a purely "physical" level (i.e., without formulating conditions of validity). The present note should be regarded, therefore, as yet another rediscovery of Feynman' s formula, but already on a "mathematical" level. For Feynman' s formula see also ([8,9]).

Before proceeding to the formulation of the result, let us agree, in order to avoid misunderstandings, on some notation.

By bold Greek letters and italic bold Latin letters we shall denote k-dimensional vectors with arbitrary components. By upright bold Latin letters we shall denote vectors with integer nonnegative components. By D_x, or by D, we shall denote symbolically the differentiation vector, i.e., $D_x = (\partial/\partial x_1, \dots, \partial/\partial x_k)$. By the product of two vectors $x = (x_1, \dots, x_k)$, $\xi = (\xi_1, \dots, \xi_k)$ we shall mean their scalar product, i.e. $x\xi = x_1\xi_1 + \cdots + x_k\xi_k$.

In the expressions $|x|$, x^m, D_x^m the following meaning will be understood:

$$|x| = |x_1| + \cdots + |x_k|, \quad x^m = x_1^{m_1} \cdots x_k^{m_k}, \quad D^m = \partial^{|\mathbf{m}|}/\partial x_1^{m_1} \cdots \partial x_k^{m_k}.$$

In addition, we shall use the following abbreviated notation for integrals:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x)\,dx_1 \cdots dx_k.$$

Let $F(x, \xi, w)$ be some function, defined and continuous for all finite real values of the variables. By the continual integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int \exp\left\{ \int_0^t F(x(s), \xi(s), x'(s))\,ds \right\} \prod d\xi(s)\,dx(s)$$

$$[x(0) = y,\ x(t) = x],$$

we shall mean the limit, as $N \to \infty$, of the expression

$$(2\pi)^{-kN}\int_{-\infty}^{\infty}\cdots\int \exp\left\{\frac{t}{N}\sum_{n=1}^{N}F\left(x_n,\xi_n,\frac{x_n-x_{n-1}}{t/N}\right)\right\}\prod_{n=1}^{N}d\xi_n\prod_{n=1}^{N-1}dx_n$$

$$[x_0=y,\ x_N=x],$$

if this limit exists (of the integrals entering the prelimit expression we require convergence, but not necessarily absolute convergence, so that the order of integration may turn out to be essential).

Theorem. Let

$$P(x,\xi)=\sum_{|\mathbf{m}|\le 2p}C_m(x)\xi^m,$$

where the coefficients $C_m(x)$ satisfy the conditions:

1. For all $\mathbf{m}$ and $\mathbf{n}$, $|\mathbf{m}|\le 2p$, $|\mathbf{n}|\le 2p$, and for all real values of x, the inequalities

$$|D^n C_m(x)|<M\exp\left[a|x|^{2p/(2p-1)}\right]$$

hold (M and a do not depend on x).

2. For any real values of x, ξ, and a the inequality holds

$$\operatorname{Re}P(x,\xi)>M_1|\xi|^{2p}.$$

Then for the function $u(x,y,t)$–the solution of the equation

$$\frac{1}{h}\frac{\partial u}{\partial t}+P(x,-iD_x)u=0,\qquad u\big|_{t=0}=\delta(x-y) \tag{2}$$

the formula holds

$$u(x,y,t)=\frac{1}{2\pi}\iint_{-\infty}^{\infty}\exp\left\{\int_0^t[-hP(x(s),\xi(s))+i\xi(s)x'(s)\,ds]\right\}\prod d\xi'(s)\,dx(s) \tag{3}$$

$$[x(0)=y,\ x(t)=x].$$

It is easy to see that the classical formula for equation (1) can be obtained from the theorem stated above by putting

$$P(x, \xi) = \xi^2 - q(x)$$

and evaluating the integrals with respect to ξ_n in the prelimit expression for the continual integral. It is interesting to note that, by performing the same operation in the case where $P(x, \xi)$ is a positive definite quadratic form in ξ with coefficients depending on x, we also obtain a continual integral with an exponential integrand. In the remaining cases, evaluation of the integrals with respect to ξ_n leads to special cases of Daletskii' s formulas.

Let us say a few words about the proof of the theorem, without carrying it out in detail.

The question of convergence of multiple integrals to the solution of a differential equation is in many respects analogous to the question of convergence of the solution of an equation according to a difference scheme to the solution of the equation as the step is refined (this is mentioned in paper ([1])). Let us trace this analogy in the case of interest to us.

Consider the operator $A(\tau)$, defined on finite functions $f(x)$ by the equality

$$[A(\tau)f](x) = \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) \exp[-\tau h P(x, \xi) + i\xi(x-y)]\, d\xi\, dy.$$

It is easy to see that

$$\left\{ \left[A\left(\frac{t}{N}\right) \right]^N f \right\}(x) = \int_{-\infty}^{\infty} f(y) K_N(x, y, t)\, dy,$$

where

$$K_N(x, y, t) =$$

$$= (2\pi)^{-kN} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ \frac{t}{N} \sum_{n=1}^{N} \left[-hP(x_n, \xi_n) + i\xi_n \frac{x_n - x_{n-1}}{t/N} \right] \right\} \prod_{n=1}^{N} d\xi_n \prod_{n=1}^{N-1} dx_n$$

$$[x_0 = y,\ x_N = x].$$

This means that the prelimit expression for the continual integral from formula (3) coincides with the result of applying the operator $[A(t/N)]^N$ to the function $\delta(x-y)$. If, however, we consider the recurrent system of operator equations

$$u_{n+1}^{(N)}(x) = A(t/N) u_n^{(N)}(x) \quad (n = 0, 1, \dots, N-1), \qquad u_0^{(N)}(x) = f(x),$$

then we also obtain

$$u_N^{(N)}(x) = [A(t/N)]^N f(x).$$

On the other hand, more or less any difference scheme for equation (2) can be written in the form of a recurrent system of operator equations

$$u_{n+1}^{(N)}(x) = B(t/N)u_n^{(N)}(x) \quad (n = 0, 1, \dots, N-1), \qquad u_0^{(N)}(x) = f(x),$$

where $B(\tau)$ is the transition operator to the next "layer," i.e., the shift operator by one step in time. The difference between the cases described is only that, for difference schemes, a "layer" is a function specified on some lattice in space, whereas for us a "layer" is a function specified at every point of space.

In the theory of difference schemes it is known that a difference scheme converges if it "approximates" the differential equation and, in addition, is "stable." The approximation condition consists in the fact that, for any finite function $f(x)$, the asymptotic relation

$$[B(\tau)f](x) = f(x) - \tau h P(x, D) f(x) + o(\tau) \qquad (\tau \to 0),$$

holds, while the necessary stability condition (very close to a sufficient one) consists in the fact that, for all real λ different from zero, the inequality

$$|B(\tau)\exp(i\lambda x)| < 1.$$

must be satisfied.

It is easily verified that, for the operator $A(\tau)$ introduced by us, both of these conditions are fulfilled.

In the very substantial article of F. John ([6]), the question of convergence of difference schemes for a parabolic equation is investigated in detail; moreover, the investigation is carried out without using the specific properties of second-order equations. In that article convergence is established not only to solutions with sufficiently smooth initial data, but also to the fundamental solution. The considerations used by John (though not the results themselves) are quite applicable to the case of interest to us. On their basis it is comparatively easy to obtain a proof of the theorem.

Let us add a few words about possible generalizations. First of all, it should be noted that the formula remains valid (without any changes) also in the case when the coefficients of the equation depend on time. In the formulation of the theorem this possibility was omitted only in order to simplify the subsequent arguments (and also to simplify the restrictions on the coefficients). Further, the very form of formula (3) suggests that it should also be valid in cases where the dependence of the function $P(x, \xi)$ on ξ is not only polynomial. Interesting results may be expected from generalizations in this direction.

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Note: Figure translations are in progress. See original paper for figures.