SOME INTEGRAL RELATIONS IN THE THEORY OF RADIATION TRANSFER

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Abstract

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SOME INTEGRAL RELATIONS IN THE THEORY OF RADIATION TRANSFER

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In Theorem 1 new integral relations are given which connect various solutions of Milne' s problem.

The mathematical problem is posed as follows. Let $g(\mu)$, $\mu \in [-1,1]$, be a scattering indicatrix, that is, a nonnegative function on $[-1,1]$ with summable square and such that

$$0 < g_0 \leqslant 1.$$

(Here and below

$$g_n = 2\pi \int_{-1}^{1} g(\mu) P_n(\mu)\, d\mu, \quad n \geqslant 0;$$

$P_n(\mu)$ is the n-th Legendre polynomial.)

Let Ω be the unit sphere of three-dimensional space, $C(\Omega)$ the space of continuous and $L_2(\Omega)$-square-summable complex-valued functions on Ω; in $L_2(\Omega)$ the scalar product is defined by

$$(f, g) = \int_{\Omega} f(\omega) \overline{g(\omega)}\, d\omega$$

and the norm by $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.

$g(\mu)$ generates a linear completely continuous operator $\hat{g} : L_2(\Omega) \to C(\Omega)$, acting according to the formula

$$\hat{g} f(\omega) \equiv (\hat{g} f)(\omega) = \int_{\Omega} g(\omega\omega') f(\omega')\, d\omega', \qquad f \in L_2(\Omega).$$

Next let $\mathcal{B}_\varepsilon$ be the class of abstract functions $x(\tau)$ of the numerical variable $\tau \in (0, \infty)$, possessing the following properties.

a) For each $\tau \in (0, \infty)$, $x(\tau)$ is an element of $L_2(\Omega)$.

b) The numerical function $\|x(\tau)\|$ is bounded on every bounded part of the half-line $\tau > 0$.

c) The abstract function $e^{-\varepsilon\tau} x(\tau)$ is summable in the sense of Bochner (^{3}) on $(0, \infty)$.

Fix some vector $\mathbf{n} \in \Omega$. Let $\varepsilon \in (-1, 1)$, $x(\cdot) \in \mathcal{B}_\varepsilon$, and

$$\hat{g}x(\tau, \omega) \equiv (\hat{g}(x(\tau)))(\omega), \quad \tau > 0, \ \omega \in \Omega.$$

It turns out that for any fixed $\omega \in \Omega$ the function $e^{-\tau}\hat{g}x(\tau, \omega)$ is summable with respect to the variable τ on $(0, \infty)$. Hence it follows that the following formula uniquely defines the operator $\hat{A}$, acting on $x(\cdot)$:

$$(\hat{A}x(\cdot))(\tau, \omega) = \begin{cases} \dfrac{1}{\omega\mathbf{n}} \displaystyle\int_0^\tau e^{-(\tau-\rho)/\omega\mathbf{n}} \, \hat{g}x(\rho, \omega) \, d\rho, & \text{for } \omega\mathbf{n} > 0, \\ \hat{g}x(\tau, \omega), & \text{for } \omega\mathbf{n} = 0, \\ \dfrac{1}{|\omega\mathbf{n}|} \displaystyle\int_\tau^\infty e^{-(\rho-\tau)/|\omega\mathbf{n}|} \, \hat{g}x(\rho, \omega) \, d\rho, & \text{for } \omega\mathbf{n} < 0. \end{cases}$$

It turns out that the classes $\mathcal{B}_\varepsilon$, $\varepsilon \in (-1, 1)$, are invariant with respect to the transformation $\hat{A}$: if $x(\cdot) \in \mathcal{B}_\varepsilon$, then also $\hat{A}x(\cdot) \in \mathcal{B}_\varepsilon$.

The Milne problem is the problem of solving the equation

$$x(\cdot) = \hat{A}x(\cdot) + f(\cdot) \tag{1}$$

in the class $\overline{\mathfrak{B}} = \bigcup_{\varepsilon \in (-1,1)} \mathfrak{B}_\varepsilon$. Here we shall be interested in the so-called Milne problem with insolation, corresponding to equation (1) with a free term f of the special form:

$$f(\tau, \omega) = B(\omega) \exp(-\tau/\omega n), \qquad \tau > 0, \quad \omega \in \Omega; \tag{2}$$

$$B \in L_2(\Omega), \qquad B(\omega) = 0 \quad \text{when } \omega n \leq 0. \tag{3}$$

We shall call problem 1 the problem (1)–(3). Problem 1 with $B \equiv 0$ is the classical homogeneous Milne problem. (Physical interpretation: problem 1 is the problem of the diffusion of radiation through the half-space $\tau > 0$ of three-dimensional space; n is the unit normal vector to the boundary of the half-space, directed inward; $B(\omega)$ is the angular distribution of the ω-radiation incident on the boundary from outside.)

In the theory of the Milne problem the characteristic equation plays a fundamental role

$$(1 + k\omega n)\psi(\omega) = g\psi(\omega). \tag{4}$$

Here k is a complex parameter, $k \in Z_0$, where Z_0 is the complex plane with cuts along the real axis from $-\infty$ to -1 and from 1 to $+\infty$ (the points ± 1 belong to the cuts).

Denote by $\mathfrak{N}$ the set of values of the parameter $k \in Z_0$ for which (4) admits a nontrivial solution $\psi \in L_2(\Omega)$. It turns out that $\mathfrak{N}$ lies entirely in the interval $(-1, 1)$ of the real axis, $\mathfrak{N}$ is symmetric with respect to the point $k = 0$, $\mathfrak{N}$ is nonempty and at most countable. If $\mathfrak{N}$ is infinite, then the points $k = \pm 1$ are limit points of $\mathfrak{N}$, and $\mathfrak{N}$ has no other limit points.

There exists exactly one $\lambda_0 \in \mathfrak{N}$, $\lambda_0 \geq 0$, such that for $k = \lambda_0$ (4) admits a sign-constant solution. Up to normalization this solution is unique. If $g_0 = 1$, then $\lambda_0 = 0$; if $g_0 < 1$, then $\lambda_0 > 0$. The interval $(-\lambda_0, \lambda_0)$ contains no points of the set $\mathfrak{N}$.

For $k \in \mathfrak{N}$ the eigenspace of (4) has finite dimension p_k, with $p_{-k} = p_k$, $k \in \mathfrak{N}$, and $p_{\lambda_0} = 1$. If $\mathfrak{N}$ contains points different from zero, then there exists a system of functions

$$\{\psi_{kp} \mid k \in \mathfrak{N},\ k \neq 0,\ p = 1, 2, \dots, p_k\} \tag{5}$$

such that

a) $\psi_{kp}(\omega)$ is real, $\psi_{kp} \in C(\Omega)$, $p = 1, 2, \dots, p_k$, $k \in \mathfrak{N}$, $k \neq 0$;

b) $((\omega n)\psi_{kp}, \psi_{kp'}) = -\delta_{p'p} \operatorname{sgn} k$ for $1 \leq p \leq p' \leq p_k$, $k \in \mathfrak{N}$, $k \neq 0$;

c) $\psi_{(-k)p}(\omega) = \psi_{kp}(-\omega)$, $\omega \in \Omega$, $p = 1 \div p_k$, $k \in \mathfrak{N}$, $k \neq 0$;

d) $\{\psi_{kp} \mid p = 1, 2, \dots, p_k\}$ is a basis of the eigenspace of equation (4) corresponding to the given value of the parameter $k \in \mathfrak{N}$, $k \neq 0$.

A system of functions (5) possessing the listed properties is not unique. We assume that, from all such systems, one has been chosen and fixed; moreover (in the case $g_0 < 1$) this has been done so that $\psi_{\lambda_0,1}(\omega) > 0$ everywhere on Ω (such a choice is possible).

Let $x(\cdot)$ be some solution of problem 1, $x(\cdot) \in \mathfrak{B}_\varepsilon$, $\varepsilon \in (-1, 1)$. Then there exists $x(0) \in L_2(\Omega)$ such that

a) $x(0, \omega) = B(\omega)$ when $\omega n > 0$;

b) $\lim\limits_{\tau=0+0} \|x(\tau) - x(0)\| = 0.$

The assertions stated above are justified in (^{1}).

We now single out two variants $1_{1,2}$ of problem 1, corresponding to two fixed functions $B = B_{1,2}$ in formulas (1)–(3). Let $\varepsilon_j \in (-1, 1)$,

and $x_j(\cdot) \in \mathcal{B}^j_\varepsilon$ are some fixed solutions of problems 1_j, $j = 1, 2$. Put

$$D = \begin{cases} \dfrac{3}{4\pi}(1 - g_1)(x_1(0), \omega n)(x_2(0), \omega n), & \text{if } g_0 = 1; \\ 0, & \text{if } g_0 < 1; \end{cases}$$

and, for each $\lambda \in \mathfrak{R}$,

$$A_\lambda = \begin{cases} \dfrac{3}{4\pi}[(x_1(0), \omega n)(x_2(0), (\omega n)^2) - (x_2(0), \omega n)(x_1(0), (\omega n)^2)], & \text{for } \lambda = 0; \\ \operatorname{sgn}\lambda \cdot \displaystyle\sum_{p=1}^{p_\lambda}(x_1(0), (\omega n)\psi_{\lambda p})(x_2(0), (\omega n)\psi_{(-\lambda)p}), & \text{for } \lambda \neq 0. \end{cases}$$

Finally, introduce the operator $\hat{\sigma} : L_2(\Omega) \to L_2(\Omega)$ by the formula $(\hat{\sigma} f)(\omega) = f(-\omega)$, $f \in L_2(\Omega)$.

Theorem 1. For all $\tau \geq 0$,

$$(x_1(\tau), (\omega n)\,\hat{\sigma}\,\overline{x_2(0)}) = D\tau + \sum_{\lambda \in \mathfrak{R} \cap (-\varepsilon_2, \varepsilon_1)} A_\lambda e^{\lambda\tau}. \tag{6}$$

(The bar over the top denotes complex conjugation; the interval $(-\varepsilon_2, \varepsilon_1)$ is the empty set if $\varepsilon_1 + \varepsilon_2 \leq 0$.)

Formula (6) makes it possible to establish certain new connections between the solutions of Milne’ s problem (both homogeneous and with insolation) and, in particular, to express explicitly the coefficients of the asymptotic expansions of the solutions of problem 1 (cf. (1)) in terms of the law of insolation B and the boundary values (for $\tau = 0$) of the solutions of the homogeneous Milne problem. Expressions of this type were first obtained by V. V. Sobolev (^{2}).

A detailed exposition of the proof of Theorem 1 and of its consequences is contained in (^{4}).

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Note: Figure translations are in progress. See original paper for figures.