DUALITY FOR COHERENT ANALYTIC SHEAVES

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Abstract

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MATHEMATICS

V. D. GOLOVIN

DUALITY FOR COHERENT ANALYTIC SHEAVES

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1. Let X be a complex analytic manifold, countable at infinity, of complex dimension n, and let F be a coherent analytic sheaf on X. Each point of the manifold X has a holomorphically complete open neighborhood U, over which there is defined an epimorphism of sheaves

$$\pi:\ O^m \to F,$$

where O is the sheaf of germs of holomorphic functions on X, and m is some positive integer. Consequently, the complex vector space $\Gamma(U;F)$ of all continuous sections of the sheaf F over U is isomorphic to the quotient space $\Gamma(U;O^m)/\Gamma(U;R)$, where R is the kernel of the epimorphism π. Endowing the space $\Gamma(U;O^m)$ with the topology of compact convergence, we thereby introduce on $\Gamma(U;F)$ a certain separated locally convex topology, which is a Fréchet topology and does not depend on the choice of the epimorphism π (cf. ([1])).

Let $\mathfrak{U} = (U_i)$ be a sufficiently fine locally finite covering of the manifold X by holomorphically complete open sets. The space $C^k(\mathfrak{U};F)$ of cochains of degree $k \geq 0$ of the covering $\mathfrak{U}$ with coefficients in F is endowed with the product topology of the spaces $\Gamma(U_{i_0}\cap...\cap U_{i_k};F)$. Further, the space $H^k(\mathfrak{U};F)$ of cohomology of the covering $\mathfrak{U}$ is endowed with the quotient-space topology $\operatorname{Ker}\delta_k/\operatorname{Im}\delta_{k-1}$, where δ_k is the (continuous) coboundary operator. Finally, the space $H^k(X;F)$ of Čech cohomology of the manifold X with coefficients in the sheaf F is endowed with the topology of the inductive limit of the spaces $H^k(\mathfrak{U};F)$ with respect to the filtering set of classes of pairwise equivalent coverings. By means of a diagram chase analogous to the proof of A. Weil' s de Rham theorem (see ([2])), it is easy to show that the canonical mapping

$$H^k(\mathfrak{U};F) \to H^k(X;F)$$

is an isomorphism of topological vector spaces.

2. Let E^k, for any integer $k \geq 0$, be the sheaf of germs of infinitely differentiable exterior differential forms of bidegree $(0, k)$ on the manifold X. Consider the complex $\Gamma(X; E^* \otimes_O F)$ of vector spaces $\Gamma(X; E^k \otimes_O F)$ $(k = 0, 1, ...)$, whose coboundary operator

$$d''_k : \Gamma(X; E^k \otimes_O F) \to \Gamma(X; E^{k+1} \otimes_O F)$$

is induced by the exterior differential $d'' : E^k \to E^{k+1}$. For each $U \in \mathfrak{U}$, endow the vector space $\Gamma(U; E^k \otimes_O F)$ with the strongest of the locally convex topologies under which the canonical bilinear mapping

$$\Gamma(U; E^k) \times \Gamma(U; F) \to \Gamma(U; E^k \otimes_O F),$$

which assigns to sections ω and ξ over U of the sheaves E^k and F, respectively, the section $x \mapsto \omega(x) \otimes \xi(x)$ of the sheaf $E^k \otimes_O F$. The space $\Gamma(X; E^k \otimes_O F)$ is endowed with the wea-

weakest of the topologies for which the restriction mappings are continuous

$$\Gamma(X; E^k \otimes_O F) \to \Gamma(U; E^k \otimes_O F) \qquad (U \in \mathfrak{u}).$$

Finally, the cohomology space $H^k(\Gamma(X; E^* \otimes_O F))$ of the complex $\Gamma(X; E^* \otimes_O F)$ is endowed with the topology of the quotient space $\operatorname{Ker} d''_k / \operatorname{Im} d''_{k-1}$.

Since E^k_x is a flat $\mathcal{O}_x$-module for every $x \in X$ (see (3)), by the Dolbeault-Grothendieck lemma the sequence of sheaves

$$0 \to F \to O_p \otimes_O F \to E^1 \otimes_O F \to \cdots$$

is exact. Consequently, by means of a diagram chase one can show that, for every $k \geq 0$, the topological vector spaces $H^k(X; F)$ and $H^k(\Gamma(X; E^* \otimes_O F))$ are canonically isomorphic.

3. Let a be a continuous linear form on the space $\Gamma(X; E^k \otimes_O F)$. If U is an arbitrary open set in X, then for any $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$ we have a section $x \mapsto \omega(x) \otimes \xi(x)$ over X of the sheaf $E^k \otimes_O F$, continuous in ω and ξ. Fixing ξ, we obtain a continuous linear form $\omega \mapsto a(\omega \otimes \xi)$ on the space $\Gamma_c(U; E^k)$ of sections with compact supports, endowed with the usual topology. Thereby a $\Gamma(U; \mathcal{O})$-linear mapping $\varphi_U : \Gamma(U; F) \to \Gamma(U; D^{n,n-k})$ is defined, where $D^{n,n-k}$ is the sheaf of germs of currents of bidegree $(n, n-k)$ on X, and

$$\langle \omega, \varphi_U(\xi) \rangle = a(\omega \otimes \xi)$$

for $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$. It is obvious that, for $V \subset U$, the mappings φ_V and φ_U satisfy the natural compatibility conditions, i.e. the family (φ_U)

determines a certain homomorphism of $\mathcal{O}$-modules $\varphi : F \to D^{n,n-k}$. Since the form a has compact support, the homomorphism φ also has compact support. Thus a canonical linear mapping of vector spaces is defined

$$\Gamma'(X; E^k \otimes_O F) \to \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k})),$$

which assigns to a continuous linear form a on $\Gamma(X; E^k \otimes_O F)$ a homomorphism $\varphi : F \to D^{n,n-k}$ with compact support.

If $\varphi = 0$, then $a = 0$. Indeed, with the help of a suitable refinement of the identity it is easy to verify that, on each compact subset of X, any global section of the sheaf $E^k \otimes_O F$ is representable as a finite sum of sections $\omega \otimes \xi$, where $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$ for some open U, and $a(\omega \otimes \xi) = \langle \omega, \varphi_U(\xi) \rangle = 0$. Consequently, the mapping $a \mapsto \varphi$ of the vector space $\Gamma'(X; E^k \otimes_O F)$ into $\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k}))$ is injective. We show that it is also surjective. Since the homomorphism φ has compact support, it suffices to define the corresponding form a on sections of the form $\omega \otimes \xi$, with $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$, by the formula $a(\omega \otimes \xi) = \langle \omega, \varphi_U(\xi) \rangle$. It is obvious that the form a so defined is continuous on $\Gamma(X; E^k \otimes_O F)$, since each of the modules $\Gamma(U; F)$ may be regarded as having a finite number of generators over the ring $\Gamma(U; \mathcal{O})$ (for which it is enough to choose U sufficiently small and holomorphically complete).

Thus the vector spaces $\Gamma'(X; E^k \otimes_O F)$ and $\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k}))$ are canonically isomorphic.

Consider the mapping

$$^t d'' : \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k})) \to \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k})),$$

identified with the mapping conjugate to

$$d'' : \Gamma(X; E^k \otimes_O F) \to \Gamma(X; E^{k+1} \otimes_O F).$$

For any sections $\omega \in \Gamma_c(U; E^k)$, $\xi \in \Gamma(U; F)$ we have:

$$\langle \omega, {}^t d'' \varphi_U(\xi) \rangle = {}^t d'' a(\omega \otimes \xi) = a(d'' \omega \otimes \xi) = (-1)^{k+1} \langle \omega, d'' \varphi_U(\xi) \rangle,$$

i.e., the mapping d'' is induced by the sheaf homomorphism

$$(-1)^{k+1} d'' : \ D^{n,n-k-1} \to D^{n,n-k}.$$

4. Let $D^{n,k}$, for $k \geq 0$, be the sheaf of germs of currents of bidegree (n, k) on the manifold X. The corresponding Dolbeault-Grothendieck resolution of the sheaf Ω^n of germs of holomorphic differential forms of degree n on X can be embedded in the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & & \Omega^n & \to & D^{n,0} & \to & D^{n,1} & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & & L^0 & \to & L^{0,0} & \to & L^{0,1} & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & & L^1 & \to & L^{1,0} & \to & L^{1,1} & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \end{array}$$

with exact rows and columns and with injective modules $L^p, L^{p,q}$ $(p, q \geq 0)$ over the sheaf of rings O. Since the sheaf F is coherent, $\mathrm{Hom}_O(F, G)_x \approx \mathrm{Hom}_{O_x}(F_x, G_x)$ for any O-module G (see (4)). On the other hand, from Malgrange' s theorem (3) it follows that $D_x^{n,k}$ is an injective O_x-module for every $x \in X$. Consequently, there is a commutative diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Gamma_c(X; \mathrm{Hom}_O(F, \Omega^n)) & \to & \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,0})) & \to & \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,1})) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Gamma_c(X; \mathrm{Hom}_O(F, L^0)) & \to & \Gamma_c(X; \mathrm{Hom}_O(F, L^{0,0})) & \to & \Gamma_c(X; \mathrm{Hom}_O(F, L^{0,1})) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Gamma_c(X; \mathrm{Hom}_O(F, L^1)) & \to & \Gamma_c(X; \mathrm{Hom}_O(F, L^{1,0})) & \to & \Gamma_c(X; \mathrm{Hom}_O(F, L^{1,1})) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \end{array}$$

in which all rows and columns, beginning with the second, are exact. By means of a diagram chase we obtain a canonical isomorphism of the cohomology vector spaces, respectively, of the complexes $\Gamma_c(X; \mathrm{Hom}_O(F, L^{*,*}))$ and $\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,*}))$.

Consider the covariant functor

$$L \to \mathrm{Hom}_{O,c}(X; F, L) = \Gamma_c(X; \mathrm{Hom}_O(F, L)),$$

defined on the category of O-modules and taking values in the category of vector spaces. We shall denote its derived functors by $\mathrm{Ext}^k_{O,c}(X; F, L)$ $(k = 0, 1, \ldots)$. Thus, it has been proved above that for every $k \geq 0$ the vector spaces $\mathrm{Ext}^k_{O,c}(X; F, \Omega^n)$ and $H^k(\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,*})))$ are canonically isomorphic.

We endow the space $\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,k}))$ with the strong topology with respect to $\Gamma(X; E^{n-k} \otimes_O F)$. Then in the space $\mathrm{Ext}^k_{O,c}(X; F, \Omega^n)$ it is natural to define the topology canonically identified with the topology of the quotient space $\mathrm{Ker}\, d''_k / \mathrm{Im}\, d''_{k-1}$, where

$$d''_k : \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,k})) \to \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,k+1}))$$

is the coboundary operator of the complex $\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,*}))$. We shall denote by $\widetilde{\mathrm{Ext}}^k_{O,c}(X; F, \Omega^n)$ the associated separated locally convex space, i.e., the quotient space of $\mathrm{Ext}^k_{O,c}(X; F, \Omega^n)$ by the closure of zero in it.

5. Since the topological vector space $H^k(X; F)$ is canonically identifiable with the quotient space $\mathrm{Ker}\, d''_k / \mathrm{Im}\, d''_{k-1}$, where

$$d''_k : \Gamma(X; E^k \otimes_O F) \to \Gamma(X; E^{k+1} \otimes_O F),$$

the dual space $(H^k(X; F))'$ may be identified with the quotient space

$$(\mathrm{Im}\, d''_{k-1})^\circ / (\mathrm{Ker}\, d''_k)^\circ,$$

where the polars are taken in

$$\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k})).$$

Since

$$\langle d''_{k-1}\omega, \varphi\rangle = \langle \omega^t, d''_{k-1}\varphi\rangle$$

for all $\omega \in \Gamma(X; E^{k-1} \otimes_O F)$, $\varphi \in \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k}))$, it follows from $\langle d''_{k-1}\omega, \varphi\rangle = 0$ (for every ω) that ${}^t d''_{k-1}\varphi = 0$, and conversely. In other words, $(\mathrm{Im}\, d''_{k-1})^\circ$ coincides with

$$\mathrm{Ker}\ {}^t d'' : \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k})) \to \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k+1})).$$

Similarly, the equality

$$\langle \omega, {}^t d''_k \varphi\rangle = 0$$

(for every φ) is equivalent to the equality $d''_k\omega = 0$, i.e. $\mathrm{Ker}\, d''_k$ coincides with

$$(d''\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k-1})))^\circ.$$

In this case, $(\mathrm{Ker}\, d''_k)^\circ$ is the closure in the weak topology of the subspace

$$d''\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k-1}))$$

in

$$\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k})).$$

Since $\Gamma(X; E^k \otimes_O F)$ is reflexive, the weak and strong closures of subspaces in

$$\Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k}))$$

coincide. Thus we have proved

Theorem 1. For $0 \le k \le n$, the vector space

$$\widetilde{\mathrm{Ext}}^{n-k}_{O,c}(X; F, \Omega^n)$$

is canonically identifiable with the space dual to the topological vector space $H^k(X; F)$.

If the mapping

$$d''_k : \Gamma(X; E^k \otimes_O F) \to \Gamma(X; E^{k+1} \otimes_O F)$$

is a homomorphism, then the dual mapping

$$^t d''_k : \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k-1})) \to \Gamma_c(X; \mathrm{Hom}_O(F, D^{n,n-k}))$$

has closed image. In this case the space

$$\widetilde{\mathrm{Ext}}^{\,n-k}_{O,c}(X; F, \Omega^n)$$

is Hausdorff and, consequently, coincides with

$$\mathrm{Ext}^{n-k}_{O,c}(X; F, \Omega^n).$$

The mapping d''_k is a homomorphism if the space $H^{k+1}(X; F)$ is finite-dimensional (see ([5])). Thus the following theorem holds; for (not necessarily) projective varieties it was first proved by Grothendieck ([6]).

Theorem 2. If the variety X is compact, then the vector space

$$\mathrm{Ext}^{n-k}_{O}(X; F, \Omega^n)$$

is canonically identifiable with the dual of the space $H^k(X; F)$.

If the sheaf F is locally isomorphic to the sheaf $\mathcal{O}^m$, then

$$\mathrm{Ext}^{n-k}_{O,c}(X; F, \Omega^n) \approx H^{n-k}_c(X; \mathrm{Hom}_O(F, \Omega^n)).$$

In particular, if $F = \Omega^p(V)$ is the sheaf of germs of holomorphic differential forms of degree p with values in the complex vector bundle V, then

$$\mathrm{Hom}_O(F, \Omega^n) \approx \Omega^{n-p}(V'),$$

where V' is the vector bundle dual to V. Consequently, we have

Theorem 3. The vector space

$$H^{n-k}_c(X; \Omega^{n-p}(V'))$$

is canonically identifiable with the dual of the topological vector space

$$H^k(X; \Omega^p(V)).$$

In the particular case when

$$d''_k : \Gamma(X; E^{p,k}(V)) \to \Gamma(X; E^{p,k+1}(V))$$

is a homomorphism, we obtain from this the result of J.-P. Serre ([5]).

Kharkov State University
named after A. M. Gorky

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CITED LITERATURE

[1] H. Cartan, J.-P. Serre, C. R., **237**, No. 2, 128 (1953).
[2] A. Weil, Comm. Math. Helv., **26**, No. 2, 119 (1952).
[3] B. Malgrange, Séminaire Schwartz, **4**, 23/1–23/11, 25/1–25/5 (1959–1960).
[4] J.-P. Serre, Ann. Math., **61**, No. 2, 197 (1955).
[5] J.-P. Serre, Comm. Math. Helv., **29**, No. 1, 9 (1955).
[6] A. Grothendieck, Séminaire Bourbaki, **9**, 149/1–149/25 (1956–1957).

Note: Figure translations are in progress. See original paper for figures.