EQUATIONS WITH SINGULAR COEFFICIENTS OF THE TYPE OF THE DELTA FUNCTION AND ITS DERIVATIVES

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.934

MATHEMATICS

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EQUATIONS WITH SINGULAR COEFFICIENTS OF THE TYPE OF THE DELTA FUNCTION AND ITS DERIVATIVES

(Presented by Academician L. I. Sedov, 19 IX 1969)

Equations with variable coefficients of the type of the delta function and its derivatives are considered, interpreted as limiting elements of weakly convergent sequences. The apparatus of such equations makes it possible to study systems whose model is based on a synthesis of discrete and continuous elements, avoiding the laborious and often practically impossible procedure of "joining." This mathematical apparatus seems to the author ideal for the study of such systems as, for example, rods, plates, and shells with discrete elements concentrated at points, on lines, and on individual surfaces: concentrated masses and moments of inertia, a reinforcing set, load-bearing layers of zero thickness, etc.

Differential equations with variable coefficients of the type of the delta function and its derivatives are contained in work (^{1}), as well as in some other works. On the basis of the Bubnov–Galerkin method, the authors reduce the solution to infinite systems of algebraic equations, which could easily have been obtained, without resorting to generalized functions, from the stationarity conditions for the energy functional. Therefore generalized functions appear in this case only as formal notation, somewhat simplifying the writing.

Below a general technique is given which makes it possible to obtain a closed-form solution of an equation with variable coefficients of the type of the delta function and its derivatives, if the solution of a certain auxiliary equation of the usual type is known.

1. Partially degenerate equations. Let T_1 and T_2 be operators with domains of definition $\mathcal{D}_1$ and $\mathcal{D}_2$.

The equation

$$Tu = F, \tag{1}$$

where $T = T_1 + T_2$, will be called partially degenerate if the inverse operator T_1^{-1} with respect to T_1 exists, and the operator T_2 is degenerate:

$$T_2 u = \Phi[x, (f_1, u), (f_2, u), \dots, (f_k, u)], \tag{2}$$

where Φ is some function of a point x of n-dimensional space and k parameters (f_l, u), $l = 1, 2, \dots, k$, which are the values of certain functionals f_l on $u(x)$.

Denote by $\mathfrak{R}$ the range of the operator T, i.e. the set of elements of the form $\{T\varphi\}$, where φ ranges over $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$. Obviously, if $F \in \mathfrak{R}$, then a solution of equation (1) exists, and this solution $u \in \mathcal{D}$.

Let $u = u^*$ be a solution of equation (1). Then

$$T_1 u^* = F - T_2 u^*.$$

Hence, taking (2) into account and denoting $(f_l, u^*) = \mu_l^*$, we have

$$u^* = T_1^{-1}[F - \Phi(x, \mu_1^*, \mu_2^*, \dots, \mu_k^*)], \tag{3}$$

and also

$$\mu_l^* = \left(f_l, T_1^{-1}\left[F - \Phi\left(x, \mu_1^*, \mu_2^*, \dots, \mu_k^*\right)\right]\right), \qquad l = 1, 2, \dots, k. \tag{4}$$

Expression (3), containing k parameters μ_l^*, and the system of k equations (4) with respect to these parameters represent a formal solution of the partially degenerate equation (1) in "mixed form." In order to obtain the solution of equation (1) in closed form, it is necessary to find a solution of the system of equations (4) that also satisfies the conditions

$$T_1^{-1}\left[F - \Phi\left(x, \mu_1^*, \mu_2^*, \dots, \mu_k^*\right)\right] \in \mathcal{Y}_2, \tag{5}$$

$$F - \Phi\left(x, \mu_1^*, \mu_2^*, \dots, \mu_k^*\right) = F^* \in \mathcal{R}_1, \tag{6}$$

where $\mathcal{R}_1$ is the range of the operator T_1.

If such a solution of the system (4) exists, then the solution of equation (1) is represented by expression (3) for the found values of the parameters μ_l^*.

An example of partially degenerate equations is provided by integral and integro-differential equations with a degenerate kernel.

2. Equations with singular coefficients.

Let

$$T_2 u = \sum_{|\alpha| \le r} \sum_{k=1}^{m} D^\alpha \delta(x - x_k) \cdot T_{2\alpha k} u. \tag{7}$$

Here $D^\alpha \delta(x - x_k)$ is the partial derivative of the δ-function of order $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, whose support is concentrated at the point $x = x_k$. Outside this point $D^\alpha \delta(x - x_k)$ is identified with the ordinary function

$$D^\alpha \delta(x - x_k) \equiv 0, \qquad x \ne x_k.$$

From the definition of the derivatives of the δ-function it follows that if $u \in \mathscr{Y}_2$, then at $x = x_k$ all derivatives $D^\alpha T_{2\alpha k} u$ exist in the ordinary (not generalized) sense.

Taking into account the local properties of the δ-function and its derivatives, for $u \in \mathscr{Y}_2$ we have

$$D^\alpha \delta(x - x_k) \cdot T_{2\alpha k} u = \sum_{i=0}^{\alpha} (-1)^{|i|} \prod_{j=1}^{n} c_{\alpha_j}^{i_j} \, D^i T_{2\alpha k} u\big|_{x = x_k} \, D^{\alpha - i} \delta(x - x_k), \tag{8}$$

where

$$\sum_{i=0}^{\alpha} = \sum_{i_1, \dots, i_n = 0}^{\alpha_1, \dots, \alpha_n}.$$

According to (8), the operator (7) is degenerate, since the quantities $D_{2\alpha k}^{iT} u\big|_{x = x_k}$ can be interpreted as values of certain singular functionals on $u(x)$:

$$D_{2\alpha k}^{iT} u\big|_{x = x_k} = (f_{\alpha i k}, u). \tag{9}$$

The equation with singular coefficients

$$T_1 u + \sum_{|\alpha| \le r} \sum_{k=1}^{m} D^\alpha \delta(x - x_k) \cdot T_{2\alpha k} u = F \tag{10}$$

on the basis of (7)—(9) is partially degenerate:

$$T_1 u + \sum_{|\alpha|\le r} \sum_{k=1}^{m} \sum_{i=0}^{\alpha} c_{\alpha i} (f_{\alpha i k}, u) D^{\alpha - i} \delta(x - x_k) = F, \tag{11}$$

where

$$c_{\alpha i} = (-1)^{|i|} \prod_{j=1}^{n} c_{\alpha_j}^{i_j}.$$

The solution of equation (11) in "mixed form" according to (3) and (4) will be

$$u = T_1^{-1} \left[F - \sum_{|\alpha|\le r} \sum_{k=1}^{m} \sum_{i=0}^{\alpha} c_{\alpha i} \mu_{\alpha i k} D^{\alpha - i} \delta(x - x_k) \right], \tag{12}$$

$$\mu_{\tilde\alpha \tilde i k} = D^{\tilde i} T_{2\tilde\alpha k} T_1^{-1} \left[F - \sum_{|\alpha|\le r} \sum_{k=1}^{m} \sum_{i=0}^{\alpha} c_{\alpha i} \mu_{\alpha i k} D^{\alpha - i} \delta(x - x_k) \right]_{x = x_k}, \tag{13}$$

$$|\tilde\alpha| \le r, \qquad k = 1, 2, \dots, m, \qquad \tilde i_j \le \tilde\alpha_j.$$

If the solution of the system (13) $\bar\mu = \bar\mu^*$, also satisfying the conditions

$$F - \sum_{|\alpha|\le r} \sum_{k=1}^{m} \sum_{i=0}^{\alpha} c_{\alpha i} \mu^*_{\alpha i k} D^{\alpha - i} \delta(x - x_k) = F^* \in \mathcal{R}_1,$$

$$T_1^{-1} F^* \in \mathcal{Y}_2,$$

exists, then expression (12) for $\bar\mu = \bar\mu^*$ represents a solution of equation (10).

Let x be a point on the line. Then

$$T_2 u = \sum_{\alpha=1}^{r} \sum_{k=1}^{m} \delta^{(\alpha)}(x - x_k) T_{2\alpha k} u.$$

If $T_1 = L_1$ is a linear differential operator, the solution of the equation

$$L_1 u = F - \sum_{\alpha=1}^{r} \sum_{k=1}^{m} \sum_{i=0}^{\alpha} c_{\alpha i} \mu_{\alpha i k} \delta^{(\alpha - i)}(x - x_k),$$

where $c_{\alpha i} = (-1)^i C_\alpha^i$, $\mu_{\alpha i k} = \dfrac{d^i}{dx^i} T_{2\alpha k} u \Big|_{x = x_k}$, can be found by the method of variation of arbitrary constants.

Consider the simplest example

$$u'' + \delta(x)u = f''(x).$$

Reducing to a partially degenerate form, we have:

$$u'' + u(0)\delta(x) = f''(x).$$

In mixed form the solution has the form

$$u(x) = c_1 x + c_2 - u(0)\dot{x} + f(x),$$

$$u(0) = c_2 + f(0),$$

where

$$\dot{x} = \begin{cases} x, & x \geq 0, \\ 0, & x \leq 0. \end{cases}$$

Eliminating $u(0)$, we finally obtain

$$u(x) = c_1 x + c_2(1 - \dot{x}) + f(x) - f(0)\dot{x}.$$

3. Conclusion. On the basis of the method set forth above, the author has considered a number of problems concerning the calculation of stiffened plates and cylindrical shells. Solutions have been obtained both for a unidirectional set, oriented in the circumferential and longitudinal directions, and for a crossed set. Beams on an elastic foundation have also been considered in the presence of

concentrated elastic constraints, vibrations of beams with concentrated masses, etc. Most of the solutions have a closed form. For equations with singular coefficients, it is possible to formulate an eigenvalue problem. The author has considered an analogue of the Sturm-Liouville problem for the equation

$$u'' + \lambda \left[1 + \mu \sum_{k=1}^{m} \delta(x - x_k)\right] u = 0$$

under the conditions $u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$

Moscow Aviation Institute
named after Sergo Ordzhonikidze

Received
10 IX 1969

CITED LITERATURE

1. D. V. Vainberg, I. Z. Roitfarb, in: *Calculation of Spatial Structures*, no. X, Moscow, 1965, p. 39.

Note: Figure translations are in progress. See original paper for figures.

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