ON THE FINALITY OF CONVERGENCE CRITERIA FOR FOURIER SERIES OF ALMOST PERIODIC FUNCTIONS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.512

MATHEMATICS

E. A. BREDIKHINA

ON THE FINALITY OF CONVERGENCE CRITERIA FOR FOURIER SERIES OF ALMOST PERIODIC FUNCTIONS

(Presented by Academician S. L. Sobolev on 10 III 1970)

1. This note gives two criteria for the uniform convergence of Fourier series of Bohr almost periodic (a.p.) functions, formulated in terms of best approximations. It is shown that, under certain restrictions imposed on the order of decrease of the best approximation, and in a definite sense specified below, these criteria are final. The theorems of this note generalize the results of paper ([1]).

2. Denote by B^* the class of Bohr a.p. functions whose Fourier exponents have a single limit point λ^* (if $\lambda^* \neq \infty$, then without loss of generality one may take $\lambda^* = 0$). Let $\{\lambda_k\}$ $(k = 1, 2, \dots;\ \lambda_k > 0)$ be a monotone sequence of absolute values of the Fourier exponents of a function $f(x) \in B^*$. We assign $f(x) \in B^*$ to the class B^*_∞ if $\lambda_k \uparrow \infty$; we assign $f(x) \in B^*$ to the class B^*_0 if $\lambda_k \downarrow 0$ and $M\{f(x)\} = 0$.

To a function $f(x) \in B^*$ there corresponds a Fourier series of the form

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x} \qquad (\lambda_0 = 0,\ \lambda_{-k} = -\lambda_k,\ A_{kA_{-k}} \neq 0 \text{ for } k \neq 0). \tag{1}$$

Put

$$E_\lambda(f) = \operatorname*{Inf}_{F(x)\in P_\lambda} \left\{ \operatorname*{Sup}_x |f(x) - F(x)| \right\},$$

$$\mathcal{E}_\lambda(f) = \operatorname*{Inf}_{F(x)\in Q_\lambda} \left\{ \operatorname*{Sup}_x |f(x) - F(x)| \right\},$$

where P_λ and Q_λ are, respectively, the classes of Bohr a.p. functions whose Fourier exponents μ_k satisfy the conditions $|\mu_k| \leq \lambda$ and $|\mu_k| \geq \lambda$.

Let $\varphi(\lambda) \uparrow \infty$ as $\lambda \to \infty$ ($\lambda \to 0$); $\varphi(\lambda)$ is continuous and $\varphi(\lambda) > 0$ for $\lambda > \lambda_0 > 0$ ($0 < \lambda < \lambda_0$). We assign $\varphi(\lambda)$ to the class $\mathcal{L}$ if there exists $\theta > 1$ such that the numerical sequence $\{\Lambda_n\}$ ($n = 1, 2, \dots;\ \lambda_n > 0$), whose terms are defined by the equalities

$$\varphi(\Lambda_n) = \theta^n \quad (n = 1, 2, \dots), \tag{2}$$

has one of the following two properties:

1) $\Phi(n, \theta) \le C(\theta)$,

(3)

where $\Phi(n, \theta) = \Lambda_n / (\Lambda_{n+1} - \Lambda_n)$ ($\Phi(n, \theta) = \Lambda_{n+1} / (\Lambda_n - \Lambda_{n+1})$), and $C(\theta)$ is a constant depending only on θ;

2) an unbounded nondecreasing function of the argument n,

$$\Phi(n, \theta) = O(a^{\theta^n}) \qquad (a > 1). \tag{4}$$

Thus, $\varphi(\lambda) \in \mathcal{L}$ if the lacunary sequence of values of this function corresponds to a lacunary (in the broad sense) sequence of values of the argument λ.

3. **Theorem 1.** Let $f(x) \in B^*_\infty$,

$$E_\lambda(f) = O[1/\varphi(\lambda)], \tag{5}$$

where $\varphi(\lambda) \uparrow \infty$ as $\lambda \to \infty$. If

$$\frac{1}{\varphi(\lambda_n)} \ln \frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1} - \lambda_n} = o(1), \tag{6}$$

then the Fourier series (1) converges uniformly to $f(x)$ on the entire real axis.

Theorem 2. Let $f(x) \in B^*_0$,

$$\mathcal{E}_\lambda(f) = O[1/\varphi(\lambda)], \tag{5'}$$

where $\varphi(\lambda) \uparrow \infty$ as $\lambda \to 0$. If

$$\frac{1}{\varphi(\lambda_n)} \ln \frac{\lambda_n + \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} = o(1), \tag{6'}$$

then the Fourier series (1) converges uniformly to $f(x)$ on the entire real axis.

Proof of Theorems 1 and 2. The uniform convergence of the Fourier series (1) of the function $f(x) \in B^*_\infty$ follows from (5), (6) and Theorem 6 of paper ([2]). The uniform convergence of the Fourier series (1) of the function $f(x) \in B^*_0$ follows from (5′), (6′) and Theorem 5 of paper ([3]).

4. The theorems given below assert that, generally speaking, in conditions (6) and (6′) of Theorems 1 and 2 one cannot replace $o(1)$ by $O(1)$.

Theorem 3. Let $\varphi(\lambda) \in \mathcal{L}$; there exists a function $f(x) \in B^*_\infty$, satisfying condition (5), such that:

1)

$$\frac{1}{\varphi(\lambda_n)} \ln \frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1} - \lambda_n} = O(1); \tag{7}$$

2) the Fourier series (1) of the function $f(x)$ diverges at the point $x = 0$.

Theorem 4. Let $\varphi(\lambda) \in \mathcal{L}$; there exists a function $f(x) \in B^*_0$, satisfying condition (5′), such that:

1)

$$\frac{1}{\varphi(\lambda_n)} \ln \frac{\lambda_n + \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} = O(1); \tag{7'}$$

2) the Fourier series (1) of the function $f(x)$ diverges at every point of the real axis.

Proof of Theorem 3. Fix $\theta > 1$, satisfying condition (2) and condition (3), or (4); then $\Lambda_{n+1}/\Lambda_n = 1 + 1/\Phi(n)$, where $\Phi(n) = \Phi(n, \theta)$.

If (3) holds, there is a $\theta_1 > 1$ such that $\Lambda_{n+1}/\Lambda_n \geq \theta_1$; in this case put

$$\varepsilon_n = \frac{1}{2} \frac{\theta_1 - 1}{\theta_1 + 1} \Lambda_n^- \quad (n = 1, 2, ...). \tag{8}$$

If (4) is satisfied, put

$$\varepsilon_n = \frac{1}{2} \frac{\Phi(1)}{1 + 2\Phi(1)} \frac{\Lambda_n}{\Phi(n)} \quad (n = 1, 2, ...). \tag{8'}$$

In consequence of (8) and (8′), the intervals $(\Lambda_n - \varepsilon_n, \Lambda_n + \varepsilon_n)$ $(n = 1, 2, ...)$ do not intersect. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{Q(x, n)}{\theta^n}, \tag{9}$$

where

$$Q(x,n) = P(x,n) - \widetilde{P}(x,n), \qquad P(x,n) = \sum_{k=1}^{N_n} \frac{1}{k} \cos\left(\Lambda_n - k\frac{\varepsilon_n}{N_n}\right) x,$$

$$\widetilde{P}(x,n) = \sum_{k=1}^{N_n} \frac{1}{k} \cos\left(\Lambda_n^- + k\frac{\varepsilon_n}{N_n}\right) x, \qquad N_n = [e^{\theta^n}].$$

There exists ([1]) a constant C such that for all x and n

$$|Q(x,n)| < C, \tag{10}$$

therefore the series (9) converges uniformly and $f(x) \in B^*_\infty$.

Fix $\lambda \geq \Lambda_1 - \varepsilon_1$; there is a natural number m such that

$$\Lambda_m - \varepsilon_m \leq \lambda < \Lambda_{m+1} - \varepsilon_{m+1}. \tag{11}$$

On the basis of (9), (11), and (10),

$$E_\lambda(f) \leq \mathrm{Sup}_x \left| f(x) - \sum_{n=1}^{m-1} \frac{Q(x,n)}{\theta^n} \right| \leq C \sum_{n=m}^{\infty} \frac{1}{\theta^n}. \tag{12}$$

From (11) and (2) we obtain

$$\varphi(\lambda) < \varphi(\Lambda_{m+1}) = \theta^{m+1}. \tag{13}$$

In view of (12) and (13), $f(x)$ satisfies condition (5).

The sequence of absolute values of the Fourier exponents of the function $f(x)$

$$\{\Lambda_n - k\varepsilon_n/N_n\} \qquad (k = N_n, N_n - 1, \dots, 1),$$

$$\{\Lambda_n + k\varepsilon_n/N_n\} \qquad (k = 1, 2, \dots, N_n)$$

$$(n = 1, 2, \dots)$$

satisfies condition (7) as a consequence of the monotone increase of $\varphi(\lambda)$ and on the basis of (2) and (8) or (8′).

The Fourier series (1) of the function $f(x)$ diverges because, in view of the inequality $P(0,n) > \ln[e^{\theta n}]$, the Cauchy convergence criterion is not fulfilled for this series at the point $x = 0$.

The proof of Theorem 4 is carried out according to the scheme of the proof of Theorem 3, with the use of Lemma 2 of work ([1]).

5. The order of decrease of the best approximations $E_\lambda(f)$ and $\mathcal{E}_\lambda(f)$ is determined by structural properties of the function $f(x) \in B^*$. In particular, the estimate

$$E_\lambda(f) = O(1/\lambda^{p+\alpha}) \qquad (p = 0, 1, 2, ...) \tag{14}$$

in the case $0 < \alpha < 1$ is necessary and sufficient ([4]) in order that, for $f(x) \in B^*_\infty$, there exist a uniformly continuous derivative of order p

$$f^{(p)}(x) \in \operatorname{Lip} \alpha. \tag{15}$$

The estimate

$$\mathcal{E}_\lambda(f) = O(\lambda^{p+\alpha}) \qquad (p = 0, 1, 2, ...) \tag{14'}$$

in the case $0 < \alpha < 1$ is necessary and sufficient ([5]) in order that, for $f(x) \in B^*_0$, there exist a primitive of order p, $f_p(x) \in B^*_0$, satisfying the condition

$$\left| \int_0^u f_p(x+u)\, du \right| = O(|u|^{1-\alpha}). \tag{15'}$$

The functions $\varphi(\lambda) = \lambda^\alpha$ $(a > 0,\ \lambda \to \infty)$, $\varphi(\lambda) = \lambda^{-\alpha}$ $(a > 0,\ \lambda \to 0)$ belong to the class $\mathcal{L}$, therefore Theorems 3 and 4 of work ([1]), as well as the assertion of their finality, are contained in the results of the present note.

The estimates (14) and (14′) for $\alpha = 1$ are necessary and sufficient for $f(x) \in B^*$ to belong to classes whose structural characteristics are given in works ([4,5]). On the basis of these structural characteristics, in view of Theorems 1–4 of the present note, one can formulate final criteria for the uniform convergence of Fourier series for $f(x) \in B^*$ under conditions imposed on $f^{(p)}_{(\infty)}$ and $f_p(x)$ that are weaker than (15) and (15′).

The author expresses gratitude to P. L. Ulyanov for posing the question and for his attention to the work.

Kuibyshev Aviation Institute
named after S. P. Korolev

Received
4 VII 1969

CITED LITERATURE

[1] E. A. Bredikhina, DAN, 171, No. 4, 774 (1966). [2] E. A. Bredikhina, Matem. sborn., 50 (92), 8, 369 (1960). [3] E. A. Bredikhina, Matem. sborn., 56 (98), 1, 59 (1962). [4] E. A. Bredikhina, Tr. Kuibyshevsk. aviatsion. inst., vol. IV, 15 (1958). [5] Chen Nai-dun, Sci. Sinica, 18, No. 2, 185 (1964).

Note: Figure translations are in progress. See original paper for figures.