ON THE PROBLEM OF CLOSURE IN THE THEORY OF TURBULENCE

HYDROMECHANICS

1970

SovietRxiv

Abstract

Full Text

UDC 532.517.45

HYDROMECHANICS

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ON THE PROBLEM OF CLOSURE IN THE THEORY OF TURBULENCE

(Presented by Academician M. D. Millionshchikov, 4 VIII 1969)

One of the schemes of approximate closure of the system of coupled equations for moments is based on the assumption that the Eulerian velocity field of a fluid is close to normally distributed. The cumulant approximation of the n-th order consists in neglecting the cumulants of moments of higher order ([1]).

For the theory of turbulence, formulated as a system of coupled equations for the joint probability distributions of the values of the velocity field at a collection of fixed points, or of other random quantities characterizing the flow ([2]), one can construct a closure method similar to the quasinormal approximation for the Friedmann–Keller system of moments.

The equation for the distribution function of n random variables usually contains, in addition to the n-th function, the distribution function of the $(n+1)$-st variable. The closure scheme for the chain of equations consists in constructing a closed system of equations for a finite number of distribution functions.

A closure variant different from ([3]) can be proposed on the basis of an exact expansion for the joint probability distribution of related variables. Namely, for the characteristic function of the distribution φ_{12} of two random variables u_1 and u_2 one may write

$$\varphi_{12}(\theta_1,\theta_2)=\varphi_1(\theta_1)\varphi_2(\theta_2)\exp\left[\sum_{k,l=1}^{\infty}\frac{S_{k,l}}{k!\,l!}(i\theta_1)^k(i\theta_2)^l\right]. \tag{1}$$

Here S_{kl} is the cumulant of the moment $\langle u_1^k u_2^l\rangle$. Expression (1) is the expansion of the characteristic function

$$\varphi_{12}=\exp\left[\sum_{k,l=0}^{\infty}\frac{S_{kl}}{k!\,l!}(i\theta_1)^k(i\theta_2)^l\right],$$

partially summed with respect to the cumulants S_{0m} and S_{q0}; here $\varphi_1(\theta_1) = \varphi_{12}(\theta_1, \theta_2 = 0)$ and $\varphi_2(\theta_2) = \varphi_{12}(\theta_1 = 0, \theta_2)$. The corresponding inversion of expansion (1) has the form

$$P_{12}(u_1, u_2) = \exp\left[\sum_{k,l=1}^{\infty} \frac{S_{kl}}{k!\,l!} \frac{\partial^{k+l}}{\partial u_1^k \partial u_2^l}\right] P_1(u_1) P_2(u_2). \tag{2}$$

The quantities

$$P_1 = \int P_{12}(u_1, u_2)\, du_2, \qquad P_2(u_2) = \int P_{12}(u_1, u_2)\, du_1$$

are the exact distribution functions of the quantities u_1 and u_2, respectively. In essence, the operator

$$\exp\left[\sum_{k,l=1}^{\infty} \frac{S_{kl}}{k!\,l!} \frac{\partial^{k+l}}{\partial u_1^k \partial u_2^l}\right]$$

is a difference integral operator. The generalization to the case of a larger number of variables is obvious.

Possible closure methods may consist in limiting the number of cumulants in (2). The system of equations for the distribution functions of velocities F_n at n fixed points can, for example, be closed if one restricts oneself to the simplest cumulants in the expansion of the distribution function F_3 of the velocities V_1, V_2, V_3 at three points x_1, x_2, x_3

$$F_3 = \exp\left[S_{11}^{\alpha\beta}(x_1, x_3) \frac{\partial^2}{\partial V_1^\alpha \partial V_3^\beta} + S_{11}^{\alpha\beta}(x_2, x_3) \frac{\partial^2}{\partial V_2^\alpha \partial V_3^\beta}\right] F_2(V_1, x_1, V_2, x_2) F_1(V_3, x_3); \tag{3}$$

here

$$S_{11}^{\alpha\beta}(x_i, x_3) = \langle V_i^\alpha V_3^\beta \rangle - \langle V_i^\alpha V_3^\beta \rangle, \quad i = 1, 2.$$

In this case, since the cumulants are completely expressed in terms of F_2 and F_1, we obtain closed equations for the latter.

Concrete proposals for truncating the cumulant series must satisfy additional conditions: preservation of normalization, symmetry (if the functions considered are symmetric with respect to permutation of some groups of arguments), incompressibility, and positive definiteness of the approximate functions introduced; for example, expression (3) for F_3 preserves the normalization of the distributions, but is not symmetric. Naturally, the closure scheme must correspond to plausible ideas about the structure of the velocity field.

An example of a physically and experimentally better substantiated method of decoupling may be the closure method based on the phenomenon of approximate statistical independence of the substantial acceleration, determined mainly by small-scale motions, from the velocity at the given point. The first equation of the chain for the joint distributions of the quantities V, $\dot{V} = A_1$, $\ddot{V} = A_2, \ldots$, where the dot denotes the total time derivative, has the following form:

$$\frac{\partial F_1}{\partial t} + V^\alpha \frac{\partial F_1}{\partial x^\alpha} + \frac{\partial}{\partial V^\alpha} \int F_2(V, A_1, x) A_1^\alpha \, dA_1 = 0. \tag{4}$$

The joint distribution of velocity and acceleration F_2 can, as was already said, be represented in the form

$$F_2 = \exp\left[\sum_{k,l=1}^{\infty} \frac{S_{kl}^{\alpha\beta}}{k!\,l!} \frac{\partial^{k+l}}{(\partial V^\alpha)^k (\partial A_1^\beta)^l}\right] F_1(V)\Phi_1(A_1),$$

where $F_1(V)$ and $\Phi_1(A_1)$ are the exact distribution functions of velocity and acceleration at the point x, and $S_{kl}^{\alpha\beta}$ are the cumulants of the corresponding correlation moments. The assumption of approximate statistical independence of velocity and acceleration can be realized by regarding the cumulants $S_{kl}^{\alpha\beta}$ as small and decreasing. Restricting ourselves to the first terms of the expansion, we find the approximate expression for F_2

$$F_2(V, A_1) = F_1(V)\Phi_1(A_1) + S_{11}^{\alpha\beta}(x) \frac{\partial F_1}{\partial V^\alpha} \frac{\partial \Phi_1}{\partial A_1^\beta}.$$

Substituting this expression into (4), we obtain the following equation for $F_1(V)$:

$$\frac{\partial F_1}{\partial t} + V^\alpha \frac{\partial F_1}{\partial x^\alpha} + S_{11}^{\alpha\beta} \frac{\partial^2 F_1}{\partial V^\alpha \partial V^\beta} + S_{01}^\alpha \frac{\partial F_1}{\partial V^\alpha} = 0, \tag{5}$$

$$S_{11}^{\alpha\beta} = \langle V^\alpha A_1^\beta \rangle - \langle V^\alpha \rangle \langle A_1^\beta \rangle, \qquad S_{01}^\alpha = \langle A_1^\alpha \rangle.$$

Equation (5) contains the unknown moment $S_{11}^{\alpha\beta}$, which is not expressed through $F_1(V)$; therefore the equation obtained is, generally speaking, not closed, and for the cumulant $S_{11}^{\alpha\beta}$, defined in the case of locally homogeneous turbulence by the dissipation $\varepsilon(x,t)$ as $S_{11}^{\alpha\beta} = \frac{1}{3}\varepsilon(x,t)\delta_{\alpha\beta}$, an additional equation is necessary.

Received
11 VI 1969

References Cited

1. M. D. Millionshchikov, DAN 32, 611 (1941).

2. F. R. Ulinich, DAN, 183, 535 (1968); A. S. Monin, PMM, 31, 1057 (1968); B. Ya. Lyubimov, DAN, 184, 1069 (1969).

3. F. R. Ulinich, B. Ya. Lyubimov, ZhETF, 55, 951 (1968).

Note: Figure translations are in progress. See original paper for figures.