# ELLIPTIC BOUNDARY-VALUE PROBLEMS DEGENERATING ON A SUBMANIFOLD OF THE BOUNDARY

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## Abstract

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# ELLIPTIC BOUNDARY-VALUE PROBLEMS DEGENERATING ON A SUBMANIFOLD OF THE BOUNDARY

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In a bounded domain $\Omega \subset R^{n+1}$ with smooth boundary $\Gamma$, consider a system of linear differential equations

$$A(x, D)u(x) = f(x), \qquad x = (x_1, \dots, x_{n+1}), \tag{1}$$

of order $\mu$ and size $q \times q$, with coefficients in $C^\infty(\Omega)$. Suppose that on $\Gamma$ the boundary condition

$$\gamma B u = g(x') \tag{2}$$

is given, where $B$ is a rectangular matrix of size $l \times q$, consisting of differential operators of order $r$; $\gamma$ is the operator of restriction of functions to $\Gamma$. It is assumed that the operator $A(x, D)$ is elliptic in $\overline{\Omega}$ and that at all points of $\Gamma \setminus \Delta$, where $\Delta$ is a smooth submanifold of dimension $n-1$ on $\Gamma$, the ellipticity condition for the problem (1), (2) is satisfied (see ([1])). Under some additional conditions, the present article proves theorems on the normal solvability of such problems. Functional spaces in which the operator corresponding to the problem is Noetherian are found.

Let a covering $\{U_i\}$ of some neighborhood of the submanifold $\Delta$ be fixed, and in each neighborhood $U_i$ let a local coordinate system (l.c.s.) $(x_1, \dots, x_{n+1})$ be so defined that $x_{n+1} > 0$ in $U_i \cap \Omega$, $\Gamma$ is given by the equation $x_{n+1} = 0$, $\Delta$ by the equations $x_{n+1} = 0,\ x_n = 0$, and for any intersecting $U_i$ and $U_j$ the transition from the l.c.s. $(x_1^i, \dots, x_{n+1}^i)$ in $U_i$ to the l.c.s. $(x_1^j, \dots, x_{n+1}^j)$ in $U_j$ is carried out by means of a transformation of the form

$$x_1^j = \varphi_1(x_1^i, \dots, x_{n-1}^i, x_{n+1}^i), \dots, x_{n-1}^j = \varphi_{n-1}(x_1^i, \dots, x_{n-1}^i, x_{n+1}^i),$$

$$x_n^j = \varphi_n(x_1^i, \dots, x_{n+1}^i), \qquad x_{n+1}^j = \varphi_{n+1}(x_1^i, \dots, x_{n-1}^i, x_{n+1}^i).$$

Suppose that in the l.c.s. corresponding to $U_i$, the operator $B$ has the form

$$B(x', D) = \sum_{j=0}^{m} \sum_{k=0}^{m-j} x_n^{\delta j} b_{jk}(x', D) D_n^k, \tag{3}$$

where $r \geq m$, $b_{jk}$ are matrix differential operators of order $r - m + j$ with coefficients in $C^\infty(R^n)$, $x' = (x_1, \dots, x_n)$, and $\delta$ is a positive integer. All that follows remains valid also in the case when

$$b_{jk}(x', D) = \sum_{l=0}^{r-m+j} b_{jk}^l(x', D') D_{n+1}^l, \tag{4}$$

where $b_{jk}^l(x', D')$ are pseudodifferential operators on the manifold $\Gamma$ of order $r - m - l + j$. Thus, for $x_n = 0$, the boundary operator $B$ degenerates into the operator $b_{0m}(x', D) D_n^m$.

For the study of problems of the indicated kind, it is natural to use weighted functional spaces. Fix some number $s$ and denote by $H^s_{(m,\delta)}(R_+^{n+1})$ the space of (generalized) functions $u(x)$ in

$R_+^{n+1} = \{x : \ x_{n+1} > 0\}$, for which the norm

$$\left\| u, \ R_+^{n+1} \right\|_s = \sum_{|\alpha| \leq m} \left\| (x_n + i x_{n+1})^{(|\alpha''| + \alpha_{n+1})\delta} D^\alpha u \right\|_s, \tag{5}$$

is finite, where $\alpha'' = (\alpha_1, \dots, \alpha_{n-1})$, $\| \ \|_s$ is the norm in the usual space $H^s(R_+^{n+1})$ (see (1)). Similarly, by $H^s_{(m,\delta)}(R^n)$ we denote the space of functions $u(x')$ in $R^n$ for which the norm

$$\left\| u, \ R^n \right\|_s = \sum_{|\alpha'| \leq m} \left\| x_n^{|\alpha'|\delta} D^{\alpha'} u \right\|_s, \tag{6}$$

is finite, where $\alpha' = (\alpha_1, \dots, \alpha_n)$, and $\| \ \|_s$ is now the norm in $H^s(R^n)$. The spaces $H^s_{(m,\delta)}(\Omega, \Delta)$ and $H^s_{(m,\delta)}(\Gamma, \Delta)$ are defined in the usual way with the aid of a partition of unity; moreover, outside a certain neighborhood of the submanifold $\Delta$, the norms in these spaces are equivalent to the usual norms in $H^{s+m}(\Omega)$ and $H^{s+m}(\Gamma)$.

The question arises of the solvability of problem (1), (2) in the spaces $u \in H^{s_0+\mu}_{(m,\delta)}(\Omega, \Delta)$, $f \in H^{s_0}_{(m,\delta)}(\Omega, \Delta)$, $g \in H^s(\Gamma)$, $s > 0$, $s_0 = s + r - \mu - m + 1/2$. As will follow from what comes later, in general problem (1), (2) has an infinite-dimensional kernel and cokernel. However, if additional boundary and coboundary conditions are prescribed on $\Delta$, then, under the fulfillment of certain algebraic requirements, the corresponding problem becomes Noetherian. Thus, let us consider on $(\Gamma, \Delta)$ boundary conditions of the form

$$\gamma Bu + G\rho(x'') \otimes \delta(\Delta) = g(x'),$$

$$\sum_{j=0}^{r-m} \gamma_\Delta C_j \gamma D_{n+1}^j u + E\rho(x'') = h(x''), \tag{7}$$

where $\gamma_\Delta$ is the operator of restriction of functions to $\Delta$, $E$ is a rectangular matrix of size $l_1 \times k$ of pseudodifferential operators on $\Delta$ with homogeneous symbols $e_{ji}(x'', \xi'')$, and $C_j$ and $G$ are pseudodifferential operators on $\Gamma$; moreover, in a neighborhood of $\Delta$ the symbol $G(x', \xi')$ for $G(x', D')$ is quasi-homogeneous of order $\sigma$, i.e. $G(x', \lambda^{1+\delta}\xi'', \lambda\xi_n) = \lambda^\sigma G(x', \xi')$, $\lambda > 0$, while the symbols $C_j(x', \xi')$ for $C_j(x', D')$ are quasi-homogeneous of order $\varkappa - (1+\delta)j$. It is assumed that $\sigma < -s - 1/2$, $\varkappa < m - 1/2$, and the order $E_{ji}$ is equal to $t - \theta$, where $\theta = (m - \varkappa - 1/2)/(1+\delta) - m + r$, $t = (\sigma + 1/2)/(1+\delta)$. The numbers $l_1$ and $k$ will be defined below.

**Theorem 1.** The operator $\mathfrak{A}:\ (u, \rho) \mapsto (f, g, h)$, which is defined by relations (1), (7), for any $s > 0$ determines a continuous mapping

$$\mathcal{H}_1^s = H^{s_0+\mu}_{(m,\delta)}(\Omega, \Delta) \times H^{t+s}(\Delta) \xrightarrow{\mathfrak{A}} H^{s_0}_{(m,\delta)}(\Omega, \Delta) \times H^s(\Gamma) \times H^{\theta+s}(\Delta) = \mathcal{H}_2^s.$$

Suppose that at each point $x_0'' \in \Delta$ the ellipticity condition for the problem

$$\gamma b_{0m}(x', D)u = \psi(x') \tag{8}$$

is fulfilled for equation (1), where $b_{0m}(x', D)$ is the operator from (3) (in the corresponding l.c.s.). A local investigation of problem (1), (7) in a neighborhood of a point $x_0'' \in \Delta$ is naturally reduced to the investigation of a certain other problem on $\Gamma$ in a neighborhood of the point $x_0''$. For this, consider a sufficiently small neighborhood $V$ of the point $x_0''$, and let $V^0 = V \cap R^n$, $R^n = \{x:\ x_{n+1} = 0\}$. Then there exists a refined regularizer of problem (1), (8), i.e. an operator

$$(f, \psi) \mapsto \mathfrak{C}(f, \psi) \quad \text{from } H^{s_0}(R_+^{n+1}) \times H^s(R_n) \text{ to } H^{s_0+\mu}(R_+^{n+1}),$$

$s > 0$, $s_0 = s + r - m - \mu + 1/2$, such that if $\operatorname{supp} f \subset V$, $\operatorname{supp} \psi \subset V^0$, then for $u = \mathfrak{C}(f, \psi)$

$$(A(x, D)u, \gamma b_{0m}(x', D)u) = (f, \psi) + T_N(f, \psi),$$

where $T_N$ is an operator smoothing by $N$ units, $N$ a sufficiently large number. Without loss of generality one may assume that $\operatorname{supp} u \subset W$, where $W$ is some large neighborhood of the point $x_0''$. Put $\mathfrak{E}_0 f = \mathfrak{E}(f,0)$, $\mathfrak{E}_1\psi = \mathfrak{E}(0,\psi)$, and introduce the operators $d_\alpha(x', D')\psi = \gamma D^\alpha \mathfrak{E}_1\psi$. As shown in [2], $d_\alpha(x', D')$ are pseudodifferential operators in $V^0$ of order $|\alpha| - (m-r)$; the symbol $d_\alpha(x', \xi')$ of the operator $d_\alpha(x', D')$ is expressed in terms of $A(x,\xi)$ and $b_{0m}(x,\xi)$. Since $u = \mathfrak{E}(f,\psi)$ satisfies equation (1) with sufficient accuracy, it remains to satisfy the boundary conditions. Substituting $u$ into the boundary conditions (7), we obtain a new problem for finding $\psi$:

$$\begin{gathered} p(x', D')\psi + G\rho(x'') \otimes \delta(x_n) = g(x') - \gamma B \mathfrak{E}_0 f = g_1(x'), \\ \gamma_\Delta C\psi + E\rho(x'') = h(x'') - \sum_{j=0}^{r-m} \gamma_\Delta C_j \gamma D_{n+1}^j \mathfrak{E}_0 f = h_1(x''), \end{gathered} \tag{9}$$

where $p(x', D')$ is obtained if in $B(x', D)$ one replaces $D^\alpha$ by $d_\alpha(x', D')$, $C = \sum C_j d_j(x', D')$, where $d_j(x', D') = \gamma D_{n+1}^j \mathfrak{E}_1\psi$. Put $C_1 = \sum C_j d_j(x', D'', 0)$, where $d_j(x', D'', 0)$ are operators with symbols $d_j(x, \xi'', 0)$, and replace $C$ in (9) by $C_1$ (the difference is then a subordinate operator). Then we obtain precisely the problem that was studied in [3].

**Theorem 2.** Suppose that problem (1), (2) outside $\Delta$ satisfies the ellipticity condition and that at each point $x_0'' \in \Delta$ the ellipticity condition is fulfilled for the corresponding problem (1), (8). Assume that the system (9), in which $C$ is replaced by $C_1$, satisfies the conditions of the paper [3]. Then the operator $\mathfrak{A}$ corresponding to problem (1), (7) is Noetherian from the space $\mathcal{H}_1^s$ into $\mathcal{H}_2^s$ for any $s > 0$.

For the proof of the theorem, right and left regularizers $R$ will be constructed. With the help of a partition of unity the construction is reduced to constructing a regularizer in a neighborhood of a point $x_0''$ of the submanifold $\Delta$, since outside $\Delta$ the ellipticity condition for problem (1), (2) is satisfied and the regularizer outside a neighborhood of $\Delta$ is constructed in the usual way. We first study the regularizer $\mathfrak{E}$ of the auxiliary problem (1), (8) in weighted spaces.

**Lemma 1.** The operator $\mathfrak{E}$ has the following property: if $f \in H^s_{(m,\delta)}(R_+^{n+1})$, $\psi \in H^s_{(m,\delta)}(R^n)$, $s > 0$, then $u = \mathfrak{E}(f,\psi)$ belongs to $H^{s+\mu}_{(m,\delta)}(R_+^{n+1})$, and

$$\| u, R_+^{n+1} \|_{s_0+\mu} \le C(\| f \|_{s_0} + \| \psi \|_s).$$

It is assumed that $\operatorname{supp} f \subset V$, $\operatorname{supp}\psi \subset V^0$. The constant $C$ does not depend on $f$ and $\psi$.

For the proof it is necessary to check the finiteness of the norms entering (5), with $s = s_0 + \mu$. We first check that $v = D_n u$ belongs to $H^{s_0+\mu}(R_+^{n+1})$. Using the conditions of the lemma, we find that $\gamma b_{0m}(x', D)v \in H^s(R^n)$, $A(x,D)v \in H^{s_0}(R_+^{n+1})$. Therefore, from the usual regularity theorem for solutions of the elliptic problem (1), (8) it follows that $v \in H^{s_0+\mu}(R_+^{n+1})$ (see, for example, ([1])). We arrive at the same conclusion if we put $v = (x_n + ix_{n+1})^\delta D_i u$, $i \neq n$. With the help of analogous arguments, we successively verify the finiteness of all norms in (5) with $s = s_0 + \mu$.

The next step consists in constructing a regularizer for system (9). In the case when $g_1(x') \in H^0(R^n)$, $h_1(x'') \in H^0(R^{n-1})$, a regularizer $S$ was constructed in ([3]) for system (9), with range in $H_{(m,\delta)}(R^n) \times H^l(R^{n-1})$.

**Lemma 2.** If the right-hand sides $(g_1, h_1)$ belong to $H^s(R^n) \times H^{0+s}(R^{n-1})$, $s > 0$, then the solution $(\psi, \rho)$ of system (9) belongs to

for $H^s_{(m,\delta)}(R^n) \times H^{l+s}(R^{n-1})$ the estimate holds

$$\left\|\psi, R^n\right\|_s + \|\rho\|_{l+s} \leq c\left(\|g_1\|_s + \|h_1\|_{\theta+s}\right).$$

It is assumed that the supports of $\psi$ and $\rho$ belong to a sufficiently small neighborhood of the point $x_0''$. The constant $c$ does not depend on $\psi$ and $\rho$.

The regularizer of the original problem (1), (7) in a neighborhood of the point $x_0'' \in \Delta$ is obtained by means of the composition of the operator $S$ and the operator $\mathfrak{C}$ (the regularizer of problem (1), (8)). Theorem 2 is proved.

As an example, consider the system

$$A(x, D)u_i(x) = f_i(x), \qquad 1 \leq i \leq q, \tag{10}$$

where $A(x, D)$ is an elliptic operator of second order, and the boundary operator (3) has the form

$$B(x', D)u = ID_{nu} + x_n^\delta b(x', D)u, \qquad u = (u_1, \dots, u_q),$$

where $b(x, D)$ is a $q \times q$ matrix of first-order differential operators, and $I$ is the identity matrix. Let $\lambda(x', \xi')$ be a root of the equation $A^0(x', \xi', \lambda) = 0$ with $\operatorname{Im}\lambda > 0$, where $A^0$ is the principal part of $A$ in the corresponding l.c.s. In this case the symbol $p(x', \xi')$ for the operator $p(x', D')$ in (9) is equal to $I\xi_n + x_n^\delta b^0(x', \xi'', \xi_n, \lambda(x', \xi'))$, $b^0$ being the principal part of $b$.

The assumptions of Theorem 2 in the present case reduce to the requirement that the matrix $b^0(x_0'', \xi'', 0, \lambda(x_0'', \xi'', 0))$ have no real eigenvalues for $\xi'' \neq 0$. If, in addition, the condition $Z_{\xi''}$ of paper (3) is satisfied, then the corresponding problem (10), (7) is Noetherian. In the case of one equation this problem (the problem with an oblique derivative) was studied in (4–8).

**Remark 1.** For simplicity of exposition we have restricted ourselves to the case where all boundary operators have the same orders and the boundary operators possess the same property. All theorems are easily generalized to the case of different orders.

**Remark 2.** We have studied the case where the order of degeneration in the boundary conditions is the same in all directions except $x_n$. Analogously to (3, 9), one can consider the case of different orders of degeneration; in particular, in some tangential directions there may be no degeneration.

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