ON THE ORDER OF APPROXIMATION OF A CONTINUOUS \(2\pi\)-PERIODIC FUNCTION BY PARTIAL SUMS OF ITS FOURIER SERIES

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Abstract

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ON THE ORDER OF APPROXIMATION OF A CONTINUOUS 2π-PERIODIC FUNCTION BY PARTIAL SUMS OF ITS FOURIER SERIES

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In this note we consider questions connected with the rate of approximation of a periodic continuous function by partial sums of its Fourier series and with the convergence of trigonometric series.

1. We adopt the following notation and assumptions. $\widetilde{C}$ is the space of real, continuous, 2π-periodic functions f with norm

$$\|f\|_{\widetilde{C}} = \|f\| = \max_{-\infty<x<\infty} |f(x)|.$$

$\widetilde{L}_q$ $(1 \le q < \infty)$ is the space of real, 2π-periodic functions summable to the q-th power on $[-\pi, \pi]$, with norm

$$\|f\|_q = \left\{ \int_{-\pi}^{\pi} |f(x)|^q \, dx \right\}^{1/q}.$$

The number p is defined by the formula $p = q/(q-1)$.

Let $f \in \widetilde{C}$. By $E_n(f)$ we denote its best approximations by trigonometric polynomials of order not exceeding n in the metric $\widetilde{C}$. Similarly, if $f \in \widetilde{L}_q$ $(1 \le q < \infty)$, then $E_n(f)_q$ denotes its best approximations in the metric $\widetilde{L}_q$. $\tilde{f}$ is the function trigonometrically conjugate to the function f. $\sigma(f)$, $S_n(f)$, $\sigma_n(f)$ are, respectively, the Fourier series of the function f, the partial sums of order n, and the Fejér sums of this series:

$$\sigma(f) = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx),$$

$$S_n(f) = S_n(f, x) = \frac{a_0(f)}{2} + \sum_{k=1}^{n} (a_k(f) \cos kx + b_k(f) \sin kx),$$

$$\sigma_n(f) = \sigma_n(f, x) = \frac{S_0(f, x) + S_1(f, x) + \cdots + S_n(f, x)}{n+1}.$$

Let $f \in \widetilde{C}$, $\rho_k(f) = \sqrt{a_k^2(f) + b_k^2(f)}$. Put

$$K_n(f)_q = \begin{cases} \sup\limits_{k \geq n} \rho_k(f), & \text{if } q = 1, \\ \left\{ \sum\limits_{l=n}^{\infty} \rho_l^p(f) \right\}^{1/p}, & \text{if } 1 < q \leq 2 \quad (n = 1, 2, \ldots), \\ \|f - S_{n-1}(f)\|_q, & \text{if } 2 < q < \infty. \end{cases}$$

It is easy to verify that

$$K_{n+1}(f)_1 \leq \pi^{-1} E_n(f)_1 \qquad (n = 0, 1, 2, \ldots).$$

By the Hausdorff-Young theorem (see (^{1}), p. 153), for $1 < q \leq 2$,

$$K_{n+1}(f)_q \leq \pi^{-1/q} \|f - S_n(f)\|_q \qquad (n = 0, 1, 2, \ldots).$$

$\omega(n, f)$ $(n = 1, 2, 3, \ldots)$ is a family of functionals defined on $\tilde{C}$ and possessing the following properties: 1) for every $f \in \tilde{C}$, $\omega(n, f) \geqslant 0$ $(n = 1, 2, 3, \ldots)$; 2) $\omega(n, f) = 0$ only for trigonometric polynomials of degree not exceeding n.

The symbol $\dfrac{0}{0}$ is everywhere understood as 0.

2. Theorem 1. Let $f \in \tilde{C}$. Then

$$\|f - S_n(f)\| \leqslant \left\{ \frac{4}{\pi^2} \ln \left(1 + n \left(\frac{K_{n+1}(f)_q}{\omega(n, f)} \right)^q \right) + 4 \right\} E_n(f) + \omega(n, f)$$
$$(n = 1, 2, \ldots),$$

if $1 \leqslant q \leqslant 2$, and

$$\|f - S_n(f)\| \leqslant \left\{ \frac{4}{\pi^2} \ln \left(1 + n \left(\frac{K_{n+1}(f)_q}{\omega(n, f)} \right)^q \right) + 4 \right\} E_n(f) + C(q)\omega(n, f)$$
$$(n = 1, 2, \ldots),$$

if $2 < q < \infty$, where the constant $C(q)$, depending only on q, satisfies the inequality

$$C(q) \leqslant \frac{\pi}{2} \left\{ \int_0^{\infty} \left(\frac{u - \sin u}{u^2} \right)^p du \right\}^{1/p} + 2.$$

This theorem strengthens the classical estimate of A. Lebesgue (see (^{2}), p. 198) and some results of Nash and Sato (see (^{3}), pp. 299–302).

Remark 1. In particular, as $\omega(n, f)$ one may take:

$$\omega(n, f) = E_n(f) \qquad (n = 1, 2, ...),$$

$$\omega(n, f) = \omega_r(1/n, f) \qquad (n = 1, 2, ...),$$

$$\omega(n, f) = \|f - U_n(f)\| \qquad (n = 1, 2, ...),$$

where $\omega_r(\delta, f)$ is the modulus of continuity of the function f of order r, and $U_n(f)$ is an approximation method assigning to each function $f \in \tilde{C}$ a trigonometric polynomial of degree not exceeding n.

Remark 2. In determining the order of approximation of a function by Fourier sums, it is sometimes useful to apply the estimates proved in Theorem 1 not to the function itself, but to a primitive of the function under consideration with the free term omitted. In this case the order of approximation of the function itself and of its conjugate is determined on the basis of the following known lemmas:

Lemma A. Let $f \in \tilde{C}$ be such that $f' \in \tilde{C}$. Then

$$\|f' - S_n(f')\| \leqslant 4E_n(f') + 10n\|f - S_n(f)\| \qquad (n = 1, 2, ...).$$

Lemma B. Let $f \in \tilde{C}$ be such that $\tilde{f}' \in \tilde{C}$. Then

$$\|\tilde{f}' - S_n(\tilde{f}')\| \leqslant 4E_n(\tilde{f}') + 10n\|f - S_n(f)\| \qquad (n = 1, 2, ...).$$

We give some consequences of Theorem 1, as well as results obtained by comparing Theorem 1 with Lemmas A and B.

Corollary 1. Let $f \in \tilde{C}$. If, for some fixed q $(1 \leqslant q < \infty)$:

1)

$$\overline{\lim}_{n\to\infty} nE_n(f)\{1 + \ln(1 + n^{1+q}K^q_{n+1}(f)_q)\} < +\infty, \tag{1}$$

then

$$\overline{\lim}_{n\to\infty} n\|f - S_n(f)\| < +\infty, \qquad \overline{\lim}_{n\to\infty} n\|\tilde{f} - S_n(\tilde{f})\| < +\infty;$$

2)

$$\lim_{n\to\infty} nE_n(f)\{1+\ln(1+n^{1+q}K^q_{n+1}(f)_q)\}=0, \tag{2}$$

then

$$\lim_{n\to\infty} n\|f-S_n(f)\|=0, \qquad \lim_{n\to\infty} n\|\tilde{f}-S_n(\tilde{f})\|=0.$$

Corollary 2. Let $f\in\widetilde{C}$ be such that $f'\in\widetilde{C}$. If, for some fixed q $(1\le q<\infty)$,

$$\lim_{n\to\infty} nE_n(f)\ln(1+nK^q_{n+1}(f')_q)=0,$$

then the Fourier series of the function f converges uniformly on the entire axis.

This criterion for uniform convergence, in particular, contains: the Hardy-Littlewood criterion (see (3), p. 271), which generalizes Jordan' s criterion; the criterion of P. Salem—S. B. Stechkin (see (3), pp. 293-296), which generalizes the Dini-Lipschitz criterion; the Sato criteria (see (3), pp. 299-302); and the corresponding part of the theorem given in (2), on p. 114, following from Lebesgue' s criterion.

Corollary 3. Let $f\in\widetilde{C}$ be such that, for some fixed q $(1\le q<\infty)$,

$$\overline{\lim}_{n\to\infty} nK^q_{n+1}(f)_q<+\infty.$$

Then there exists a constant C, independent of n, such that

$$\|f-S_n(f)\|\le CE_n(f)\{1+|\ln E_n(f)|\} \qquad (n=1,2,\ldots).$$

Corollary 4. Let $f\in\widetilde{C}$ be such that: 1) $f\in\mathrm{Lip}\,1$; 2) for some fixed q $(1\le q<\infty)$, (1) is satisfied.

Then there exists a constant C, independent of n, such that

$$\|S_n(f')\|\le C \qquad (n=1,2,\ldots).$$

Theorem 2. Let $f\in\widetilde{L}_q$ $(1<q<\infty)$ be such that

$$\overline{\lim}_{n\to\infty} n^{1/q}E_n(f)_q<+\infty.$$

Then, at the point x, the relations

$$\lim_{n\to\infty} \sigma_n(f,x)=f(x), \tag{3}$$

$$\lim_{h\to 0}\frac{1}{2h}\int_{-h}^{h} f(x+t)\,dt = f(x), \tag{4}$$

$$\lim_{n\to\infty} S_n(f,x) = f(x) \tag{5}$$

are equivalent, i.e., the fulfillment of one of the relations (3)–(5) entails the fulfillment of the other two.

Remark. The special case of the theorem, when

$$\overline{\lim}_{n\to\infty} n\rho_n(f) < +\infty,$$

was established by Hardy and Littlewood (see (3), p. 271; (2), p. 133).

Theorem 3. Let the function $f \in \widetilde{C}$.

1) If

$$\lim_{n\to\infty} n\|f - S_n(f)\| = 0, \tag{6}$$

then

$$\lim_{n\to\infty}\left\| S_n'(f,x) - \frac{n\left\{f\left(x+\frac{1}{n}\right) - f\left(x-\frac{1}{n}\right)\right\}}{2}\right\| = 0.$$

2) If

$$\lim_{n\to\infty} n\|f - S_n(f)\| < +\infty, \tag{7}$$

then:

a) the relation

$$\lim_{n\to\infty} S_n'(f,x) = S < +\infty$$

entails the relation

$$\lim_{h\to 0}\left\{\frac{f(x+h) - f(x-h)}{2h}\right\} = S;$$

b) under the condition that $f \in \mathrm{Lip}\,1$, for every function $g \in L$ Parseval' s formula is valid:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)f'(x)\,dx = \sum_{n=1}^{\infty}(a_n(g)a_n(f') + b_n(g)b_n(f')).$$

The first assertion of Theorem 3 strengthens the theorem given in (^{2}) on pp. 505-506 (see also (^{3}), pp. 183-185). Assertion a) of the second part of the theorem strengthens the corresponding result from the theorem given in (^{2}) on pp. 505-506, and combines it with the corresponding part of the theorem placed in (^{2}) on pp. 506-507. Assertion b) strengthens the results concerning Parseval' s formula given in (^{3}) on pp. 220-221.

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Note: Figure translations are in progress. See original paper for figures.

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