PERIODIC CONVOLUTION EQUATIONS AND PROPERTIES OF THEIR SOLUTIONS

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Abstract

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PERIODIC CONVOLUTION EQUATIONS AND PROPERTIES OF THEIR SOLUTIONS

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By a periodic convolution equation we shall mean an equation of the form

$$\int_a^b k(x-\xi)q(\xi)\,d\xi = 2\pi\varphi(x), \qquad a \le x \le b, \quad \sigma = b-a < T, \tag{1}$$

where $k(t)$, $\varphi(t)$ are periodic functions with period T. Many mixed problems of the theory of elasticity are reduced to the indicated equation ($^{1-3}$). In particular, contact problems of the theory of elasticity for bodies of finite dimensions are reduced to them (^{2,4}). In applications the most frequent cases are those in which $k(t)$ has a logarithmic singularity at zero, which makes it possible to investigate equation (1) by separating off the logarithmic part (^{2}). However, it is easily seen that this technique makes it possible to study equation (1) with sufficient effectiveness under the assumption that the quantity σ is small. Equation (1) can also be reduced to the so-called problem of "paired series" (^{3,4}), but here too the investigation is carried out effectively under the above-mentioned assumption.

In the present note we give the result of the investigation of equation (1) under the assumption that σ is not a small quantity. The investigation is carried out for the kernels most frequently encountered in applications and is based on reducing equation (1) to an infinite system of linear algebraic equations. The infinite system proves to be best conditioned if $2\sigma = T$; in a neighborhood of this value of the parameter it becomes possible to construct an asymptotic expansion of the solution of equation (1).

1°. Below we shall assume that the kernel of equation (1) has the form

$$k(t) = \sum_{n=-\infty}^{\infty} K(n)\exp(2\pi itn/T). \tag{2}$$

Let us introduce some notation. Suppose that the representation

$$\varphi(x) = T^{-1} \sum_{n=-\infty}^{\infty} \Phi(n) \exp(2\pi i n x T^{-1}) \tag{3}$$

holds.

By $H(\lambda)$ we shall denote the space of functions possessing the property

$$\sum_{n=-\infty}^{\infty} |n|^{\lambda} |\Phi(n)|^2 < \infty. \tag{4}$$

By $c_k^{\lambda}(a,b)$ we shall denote the space of functions specified on $[a,b]$ whose derivative of order k satisfies the Hölder condition with exponent λ; $c(a,b)$ is the space of functions continuous on $[a,b]$.

By $S(\sigma)$ we shall denote the space of complex sequences $\{x_n\}$ converging to zero with weight, i.e. such that

$$\sup_n |n^{\sigma} x_n| < \infty, \qquad \lim_{n\to\infty} n^{\sigma} x_n = 0.$$

Theorem 1. Let the following conditions hold:

1. $K(n) > 0,\ |n| < \infty.$
2. $K(n) = c|n|^{-1}[1 + o(1)],\ n \to \infty.$
3. $\varphi(x) \in H(1).$

In this case equation (1) is uniquely solvable in $H(-1)$.

Lemma 1. The embedding $H(-1) \supset L_p(a,b),\ 1 < p < 2$, holds.

Theorem 2. Let the following conditions hold:

1. $K(n) > 0,\ |n| < \infty.$
2. $K(n) = c|n|^{-1}[1 + \alpha|n|^{-1} + \beta|n|^{-2} + O(n^{-3})],\ |n| \to \infty.$
3. $\varphi(x) \in C_1^{\lambda}(a,b),\ \lambda > 0.5.$

Then equation (1) is uniquely solvable in the class of functions

$$q(x)[(b-x)(x-a)]^{0.5} \in C(a,b). \tag{5}$$

Theorem 3. Let the following conditions hold:

1. $K(z)$ is an even function, real on the real axis, meromorphic in the complex plane, representable as the ratio of two entire functions whose growth indicators have the value $\lambda|\cos\varphi|,\ \lambda > 0.$
2. $K(z) = cz^{-1}[1 + \alpha z^{-1} + O(z^{-2})],\ z \to \infty.$
3. $K(z) > 0,\ |z| < \infty,\ \operatorname{Im} z = 0.$

4. The zeros z_n and poles ζ_n of the function $K(z)$ are simple and have the asymptotics

$$z_n, \zeta_n = \pm in[1 + O(n^{-1} \ln n)], \ n \to \infty. \tag{6}$$

The stated conditions are sufficient for the representation

$$K(z) = K_+(z) K_-(z), \qquad K_+(z) \sim (cz)^{-0.5}, \quad z \to \infty, \tag{7}$$

where $K_+(z)$ and $K_-(z)$ are functions regular respectively in the upper and lower half-planes and having no zeros in the indicated domains.

5. $K(z)$ tends to 0 on a system of "proper circles" (see, for example, (5), p. 421).
6. $K'_+(-z_n) \approx An^{-0.5}$; $[K_-^{-1}(\zeta_n)]' \sim Bn^{0.5}$, $n \to \infty$.

Then the unique solution in $L_p(a, b)$, $p > 1$, of equation (1), possessing property (5) in the notation (3), is given by the relation

$$q(\xi) = \sum_{n=-\infty}^{\infty} \Phi(n) \left\{ K^{-1}(n) \exp \varkappa in\xi + \sum_{l=1}^{\infty} [A_l(n) \exp \varkappa i z_l (\xi - a) + \right.$$

$$\left. + B_l(n) \exp \varkappa i z_l (b - \xi)] \right\}, \qquad \varkappa = 2\pi/T, \quad \operatorname{Im} z_l > 0. \tag{8}$$

The coefficients $A_l(n), B_l(n)$ are determined from the infinite system of equations

$$\sum_{l=1}^{\infty} \left\{ \frac{1 - \exp \varkappa i[z_l \sigma + \zeta_r (T - \sigma)]}{\zeta_r - z_l} \pm \frac{\exp \varkappa i z_l \sigma - \exp \varkappa i \zeta_r (T - \sigma)}{\zeta_r + z_l} \right\} X_l^{\pm} =$$

$$= \frac{\exp \varkappa ina - \exp \varkappa i[bn + \zeta_r (T - \sigma)]}{(n - \zeta_r) K(n)} \pm \frac{\exp \varkappa inb - \exp \varkappa i[na + \zeta_r (T - \sigma)]}{(n + \zeta_r) K(n)}, \tag{9}$$

$$r = 1, 2, \dots; \qquad n = 0, \pm 1, \pm 2, \dots;$$

$$A_l(n) = 0.5(X_l^+ + X_l^-); \qquad B_l(n) = 0.5(X_l^+ - X_l^-);$$

$$\operatorname{Im} z_l > 0; \qquad \operatorname{Im} \zeta_r > 0.$$

For each n, the infinite system is uniquely solvable in $S(\gamma)$ for any $\gamma < 0.5$.

2°. Let us dwell on methods for constructing the solution of system (9). The coefficients of the system, along with the component $(\zeta_r - z_l)^{-1}$, contain components which, by virtue of property (6), decrease exponentially as the number grows. Therefore, under certain conditions, one can construct the exact solution of the system by the method of paper (6a).

To construct an approximate solution of the system, one may use Lemma 1 of paper $^{(6b)}$ and Lemma 4.1 of paper $^{(6v)}$. The approximate solution will be expanded in the parameters $\exp \varkappa i z_l \sigma$, $\exp \varkappa i \zeta_r (T - \sigma)$.

In solving contact problems it was found that, in nonperiodic convolution equations, the zero term of the expansion indicated above $^{(7)}$ is very effective. It corresponds to the case when, in the coefficients of a system of the form (9), the exponential components $^{(6a)}$ are absent; this can be achieved by a certain natural limiting passage with respect to the parameter. The infinite system free of exponential perturbations is equivalent to a convolution equation on the half-axis $^{(6b)}$.

In the case of periodic convolution equations, as is seen from (9), for no natural values of the parameters of the problem does one obtain a system free of exponential perturbations.

The system (9) will, evidently, be closest to the unperturbed one (the best conditioned) when the parameters satisfy the relation $2\sigma = T$; here it is necessary to take into account the same order of the zeros and poles of the function $K(z)$ (6).

Let us write the solution of the system (9) in the zero approximation. The system corresponding to this solution obviously has the form

$$\sum_{l=1}^{\infty} (\zeta_r - z_l)^{-1} X_l^{\pm}(0) = [(n - \zeta_r) K(n)]^{-1} \exp \varkappa i n a \mp [(n + \zeta_r) K(n)]^{-1} \exp \varkappa i n b. \tag{10}$$

The latter is easily solved by the method of paper $^{(6b)}$. Solving it and computing (8), we obtain an approximate solution of equation (1).

3°. As an example, let us consider the case of the right-hand side $\varphi(x) = \exp \varkappa i n x$, and let the interval on which it is prescribed be the segment $[-a, a]$. The approximate solution of equation (1) for this case can, by solving system (10), be represented in the form

$$\begin{aligned} q(x) = K^{-1}(n) \exp \varkappa i n x - f(in, a + x) - f(-in, a - x) + \\ + O[\exp \varkappa i z_1 2a,\ \exp \varkappa i \zeta_1 (T - 2a)]. \end{aligned} \tag{11}$$

Here

$$f(in,t) = \frac{1}{2\pi i K(n) \exp \varkappa ina} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{e^{-izt}\, dz}{K_+(z)(z+n)}, \qquad 0 < \varepsilon < \operatorname{Im} z_1.$$

Comparing (11) with formula (3.1) of paper (6b), we conclude that (11) is the zero term of the asymptotics, as $\sigma \to \infty$, of the nonperiodic convolution equation with kernel

$$k(t) = \int_{-\infty}^{\infty} K(z) e^{izt} dz. \tag{12}$$

Thus, if in the periodic convolution equation (1) with kernel (2) the parameters are such that $2\sigma \sim T$, then as its approximate solution one may take the zero term of the asymptotics, as $\sigma \to \infty$, of the solution of equation (1) with kernel (12).

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CITED LITERATURE

[1] Yu. I. Cherskii, DAN, **140**, No. 1 (1961).
[2] V. M. Aleksandrov, A. V. Belokon, PMM, **31**, issue 4 (1967).
[3] A. A. Babloyan, PMM, **31**, issue 4 (1967).
[4] A. A. Babloyan, N. O. Gulkonyan, Izv. AN ArmSSR, Mekh., **22**, No. 1 (1969).
[5] M. A. Lavrent' ev, B. V. Shabat, *Methods of the Theory of Functions of a Complex Variable*, "Nauka," 1965.
[6] V. A. Babeshko, a) PMM, **30**, issue 4 (1966); b) PMM, **31**, issue 1 (1967); c) PMM, **33**, issue 1 (1969).
[7] V. M. Aleksandrov, V. A. Babeshko, Izv. AN SSSR, Mekh., issue 2, 95 (1965).

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