ON FREE PRODUCTS AND ALGORITHMIC PROBLEMS IN R-VARIETIES OF UNIVERSAL ALGEBRAS

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ON FREE PRODUCTS AND ALGORITHMIC PROBLEMS IN R-VARIETIES OF UNIVERSAL ALGEBRAS

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In 1951 Evans ([7]) introduced the concept of a partial algebra in a primitive class of universal algebras and proved a general assertion on the solvability of the word identity problem for finitely defined algebras of any primitive class in which the embedding theorem for any finite partial algebra in an algebra holds. In a number of papers various authors have used the embedding theorem to solve other algorithmic problems ([2,5,6]), as well as to prove certain propositions on free algebras and free products of algebras ([1–5,8]).

In the present note we describe the so-called condition R (stronger than the embedding theorem), under which in a variety of universal algebras the algorithmic problems of word identity, isomorphism, and occurrence are positively solvable, and there also hold theorems on free algebras and free products of algebras analogous to the Nielsen–Schreier, Kurosh, and Grushko theorems from group theory. We note that the family of R-varieties is sufficiently broad; in particular, it contains all varieties of algebras considered in the above-mentioned papers ([1–6,8]).

All R-varieties of quasigroups and loops have been described by us by means of systems of identities.

Let Ω be any set of operations, consisting of systems Ω_0 (possibly $\Omega_0 = \emptyset$) of 0-ary operations and Ω_1 of operations f_i of arities $n_i > 0$, $i \in I$. A variety, or primitive class, of universal algebras with set of operations Ω and with system of identities Σ will be denoted by $\mathfrak{A}(\Omega, \Sigma)$, and each algebra from $\mathfrak{A}(\Omega, \Sigma)$ will be called an $\mathfrak{A}$-algebra. The set of all partial $\mathfrak{A}$-algebras in the sense of Evans ([7]) will be denoted by $\mathfrak{A}_r(\Omega, \Sigma)$. Here we assume that in each partial $\mathfrak{A}$-algebra A all operations from Ω_0 are defined, i.e. $\Omega_0 \subseteq A$. In what follows, as a rule, the class $\mathfrak{A}(\Omega, \Sigma)$ will be fixed and all algebras under consideration will belong to $\mathfrak{A}(\Omega, \Sigma)$. Therefore, for brevity, when writing the word $\mathfrak{A}$-algebra we shall agree to omit the letter $\mathfrak{A}$.

Let $A \in \mathfrak{A}(\Omega, \Sigma)$ and $f_\alpha \in \Omega_1$. A relation of the form

$$f_\alpha(a_{i_1}, \dots, a_{i_{n_\alpha}}) = a \tag{1}$$

for elements $a_{i_1}, \dots, a_{i_{n_\alpha}}, a \in A$, will be called tabular. A tabular relation that is identically satisfied in any partial algebra from $\mathfrak{A}_r(\Omega, \Sigma)$ will be called trivial. The set of all tabular relations holding in a partial algebra A will be denoted by $S(A)$.

A partial algebra A_1 is called a free extension of a partial algebra A if there exists a homomorphism $\varphi : A \to A_1$ satisfying the conditions:

1. A_1 is generated by the elements $\varphi(a)$, $a \in A$.
2. For any homomorphism f of A into an algebra B there exists a homomorphism $\psi : A_1 \to B$ such that $f = \psi\varphi$.

A free extension for A that is an algebra is called the free closure of A and is denoted by $\bar{A}$ (1). It is not hard to establish that the free closure of a partial algebra A is the algebra A^* given by the set of generators A and the system of defining relations $S(A)$. Every other free closure of A is isomorphic to A^*. Therefore, in what follows we shall assume that $\bar{A} = A^*$.

In the particular case where the partial algebra A is the set-theoretic union of algebras $A_i,\ i \in K$, such that $A_i \cap A_j = \Omega_0$ for $i \neq j$, the free closure coincides with the free product

$$\prod_{i\in K}^{*} A_i, \quad \text{i.e.} \quad \prod_{i\in K}^{*} A_i = \overline{\bigcup_{i\in K} A_i}.$$

A partial algebra A is called embeddable in a partial algebra A_1 if there exists a one-to-one mapping $\varphi : A \to A_1$ such that, for any $f_\alpha \in \Omega_1$ and $a_{i_1}, \dots, a_{i_{n_\alpha}}, a \in A$, the relation $f_\alpha(a_{i_1}, \dots, a_{i_{n_\alpha}}) = a$ holds in A if and only if $f_\alpha(\varphi(a_{i_1}), \dots, \varphi(a_{i_{n_\alpha}})) = \varphi(a)$ in A_1. If every partial algebra is embeddable in an algebra, then one says that the embedding theorem holds in the class $\mathfrak{A}(\Omega, \Sigma)$.

Below in this note, without further qualification, we shall consider only such varieties of algebras in which the embedding theorem holds. In this connection, without loss of generality, one may assume that every partial algebra is contained as a partial subalgebra in any of its free extensions.

We shall call a partial algebra A_0 a base of a partial algebra A if A is a free extension of A_0 and A_0 is not a free extension of any of its own proper partial subalgebras. In this case the partial subalgebra A_0', consisting of all elements connected in A_0 only by trivial relations, will be called the free part of the base A_0, and the subalgebra A_0'' of the remaining elements of A_0—the connected part of the base A_0.

Let $A \in \mathfrak{A}_r(\Omega, \Sigma)$; let $a_{i_1}, \dots, a_{i_{n_\alpha}} \in A$, and suppose the word $f_\alpha(a_{i_1}, \dots, a_{i_{n_\alpha}})$ is not defined in A. The minimal free extension A_1 of the partial algebra A in which the word $f_\alpha(a_{i_1}, \dots, a_{i_{n_\alpha}})$ is defined will be called a simple free extension of A. If in A_1

$$f_\alpha(a_{i_1}, \dots, a_{i_{n_\alpha}}) = a, \tag{2}$$

then we shall write

$$A_1 = [A;\ f_\alpha(a_{i_1}, \dots, a_{i_{n_\alpha}}) = a]. \tag{3}$$

Thus, a simple free extension of a partial algebra A is obtained from it by adjoining one new element with a relation of the form (2) and with all tabular relations that can be obtained by transformations using the adjoined relation, the relations from $S(A)$, and the identities from Σ.

We shall say that the system of identities Σ satisfies condition R if:

1. In the class $\mathfrak{A}_r(\Omega, \Sigma)$ the embedding theorem holds.
2. Whatever the partial algebra $A \in \mathfrak{A}_r(\Omega, \Sigma)$ and its simple free extension (3), every relation from $S(A_1) \setminus S(A)$ is a consequence of relation (2) and the system of identities Σ, and, if in $S(A_1) \setminus S(A)$ there is a relation

$$f_\beta(b_{j_1}, \dots, b_{j_{n_\beta}}) = b, \tag{4}$$

in which the adjoined element a occurs, then a is expressed by a word of rank 1 through the other letters contained in relation (2). (By the rank of a word we mean the total number of operations from Ω_1 occurring in this word.)

A variety of universal algebras that can be specified by a system of identities satisfying condition R will be called an R-variety.

If the system of identities Σ satisfies condition R, then for algebras from the variety $\mathfrak{A}(\Omega, \Sigma)$ the following assertions hold.

Theorem 1. If $A \in \mathfrak{A}_r(\Omega, \Sigma)$ and B is a subalgebra of the algebra $\overline{A}$, then B is the free product of the algebra $\overline{A \cap B}$ and of some free algebra $F \in \mathfrak{A}(\Omega, \Sigma)$.

Corollary 1. If an algebra A from $\mathfrak{A}(\Omega, \Sigma)$ decomposes into the free product $\prod^*_{i \in K} A_i$, and B is a subalgebra in A, then

$$B \cong \prod^*_{i \in K} (A_i \cap B) * F,$$

where F is a free algebra.

Corollary 2. Every subalgebra of a free algebra from $\mathfrak{A}(\Omega, \Sigma)$ is free.

Theorem 2. Let A be a finitely generated partial algebra from $\mathfrak{A}_r(\Omega, \Sigma)$, and let $\{a_1, \dots, a_n\}$ be a system of generators of the algebra $\overline{A}$. Then there exists a system of generators of the algebra $\overline{A}$, consisting of n elements and contained in A.

Corollary 1. The minimal number of generators of the algebra $\overline{A}$ is equal to the minimal number of generators of the partial algebra A.

Corollary 2. The minimal number of generators of a finitely generated algebra from $\mathfrak{A}(\Omega, \Sigma)$ is equal to the sum of the corresponding numbers for all factors of any of the free decompositions of this algebra.

If the system of operations Ω is finite, then for varieties of universal algebras $\mathfrak{A}(\Omega, \Sigma)$ with a finite system of identities Σ satisfying condition R, the following assertions can be proved.

Theorem 3. Any two bases of a finitely presented algebra from $\mathfrak{A}(\Omega, \Sigma)$ are isomorphic, and their bound parts coincide.

Corollary 1. For finitely presented algebras from the class $\mathfrak{A}(\Omega, \Sigma)$, the isomorphism problem is solvable.

Corollary 2. The automorphism group of a finitely presented algebra from $\mathfrak{A}(\Omega, \Sigma)$ decomposes into the direct sum of some finite group and the automorphism group of a free algebra of finite rank.

Theorem 4. If $A \in \mathfrak{A}_r(\Omega, \Sigma)$ and $a_1, \dots, a_n, a \in A$, then the element a is expressed in the algebra $\overline{A}$ by a word in the alphabet $\{a_1, \dots, a_n\}$ if and only if a is expressed by some word in the alphabet $\{a_1, \dots, a_n\}$ in the partial algebra A.

Corollary. For finitely presented algebras from $\mathfrak{A}(\Omega, \Sigma)$, the embedding problem is solvable.

The following theorem describes, in terms of identity relations, all R-varieties of quasigroups.

The class $\mathfrak{A}(\Omega, \Sigma)$ will be a variety of quasigroups if Ω consists of three binary operations $\cdot, /, \backslash$, and Σ contains the subsystem of identities Σ_0:

$$(xy)/y = x, \qquad x\backslash(xy) = y, \qquad (x/y)y = x, \qquad x(x\backslash y) = y.$$

Theorem 5. The class of quasigroups $\mathfrak{A}(\Omega, \Sigma)$ is an R-variety if and only if $\Sigma = \Sigma_0 \cup \Sigma_1$, where Σ_1 is equivalent to some subsystem of identities:

$$\begin{array}{rrr}
xy = yx; & (xy)y = x; & x(xy) = y; \\
y(xy) = y; & (xy)x = y; & \bar{x}x = x; \\
(xx)x = x; & x(xx) = x; & (xx)(xx) = x; \\
(xx)(xx) = xx; & (x\backslash x)x = x\backslash x; & x\backslash x = x/x; \\
(x\backslash x)(x\backslash x) = x\backslash x; & x(x/x) = x/x; & (x/x)(x/x) = x/x.
\end{array}$$

In the theory of quasigroups an important place is occupied by loops (quasigroups with identity e), in which the right inverse element a^{-1} for a coincides with the left inverse ${}^{-1}a$. We shall consider such loops as universal algebras with three binary operations $(\cdot)$, $(\backslash)$, $(/)$, a nullary

by the operation e and the unary operation ${}^{-1}$, and in defining a partial loop we require that in the partial loop, together with every element a, the element a^{-1} be contained. Then the following can be proved.

Theorem 6. The class of loops $\mathfrak{A}(\Omega, \Sigma)$ (with the condition $x^{-1} = {}^{-1}x$) is an R-variety if and only if $\Sigma_i = \Sigma_0' \cup \Sigma_1'$, where Σ_0' is obtained by adjoining to Σ_0 the identities

$$e/x = x^{-1}, \qquad x\backslash e = x^{-1},$$

and Σ_1' is some nontrivial subsystem of the identities:

$$\begin{array}{lll} xy = yx; & (xy)y = x; & x(xy) = y; \\ x(yx) = y; & xx = e; & x^{-1}y^{-1} = (xy)^{-1}; \\ x^{-1}y^{-1} = (yx)^{-1}; & (xy)x^{-1} = y; & x^{-1}(xy) = y; \\ (xy)y^{-1} = x; & (xy)^{-1}x = y^{-1}; & x(xy)^{-1} = y^{-1}; \\ (xy)^{-1}y = x^{-1}; & x(xx) = x^{-1}; & (xx)x = x^{-1}; \end{array}$$

$$\begin{array}{lll} x(xx)^{-1} = x^{-1}; & x^{-1}(xx) = x; & x^{-1}(xx)^{-1} = x; \\ (xx)^{-1}x = x^{-1}; & (xx)x^{-1} = x; & (xx)^{-1}x^{-1} = x; \\ xx = (xx)^{-1}; & (xx)^{-1} = x^{-1}x^{-1}; & xx = x^{-1}x^{-1}; \\ (xx)(xx) = (xx)^{-1}; & (xx)^{-1}(xx)^{-1} = xx; & x(xx) = (xx)^{-1}; \\ (xx)x = (xx)^{-1}; & (xx)x^{-1} = (xx)^{-1}; & (xx)^{-1}x = xx; \\ (xx)^{-1}x^{-1} = xx; & (xx)(xx) = x; & (xx)^{-1}(xx)^{-1} = x; \\ (xx)(xx) = x^{-1}; & (xx)^{-1}(xx)^{-1} = x^{-1}; & (xx)^{-1} = x; \\ x^{-1}(xx) = (xx)^{-1}; & x^{-1}(xx)^{-1} = xx. & \end{array}$$

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Note: Figure translations are in progress. See original paper for figures.