ON THE STABILITY OF THREE-LEVEL DIFFERENCE SCHEMES

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Abstract

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Corresponding Member of the USSR Academy of Sciences A. A. SAMARSKII

ON THE STABILITY OF THREE-LEVEL DIFFERENCE SCHEMES

1. The general theory of stability of three-level schemes in the Hilbert space H_h was studied in ([1],[3]). In ([1]), a priori estimates were obtained expressing the stability of operator-difference three-level schemes in special (compound) norms containing the values of the solution on two neighboring layers. In ([3]), the necessity of the stability condition $\hat{R} > {}^1/_4 A$, obtained in ([1]), was shown.

Here it will be shown that from the results of ([1]) there follow a priori estimates for the solution in the usual energy norms $\bar{H}_A$ and $\bar{H}_B$ in terms of the right-hand sides in $H_{B^{-1}}$ and $H_{A^{-1}}$ (A and B are the operators of the three-level scheme, see (1)). A number of new a priori estimates is obtained.

2. We shall use the notation of ([2]): H_h is a real Hilbert space (with scalar product (,) and norm $\|x\| = \sqrt{(x,x)}$), depending on the parameter h, an element of some normed space; $\omega_\tau = \{t_k = k\tau,\ k = 0, 1, \ldots, k_0\}$ is a mesh with step τ on the interval $0 \le t \le t_0$; $y_{h\tau}(t_k)$, $\varphi_{h\tau}(t_k)$, etc. are abstract functions of the argument $t_k \in \omega_\tau$ with values in H_h; $A_{h\tau}(t_k)$, $B_{h\tau}(t_k)$, etc. are linear operators specified on H_h. The indices h, τ are, as a rule, omitted.

The notation $A = A^* > 0$ means that A is a self-adjoint operator and $(Ax, x) > 0$, $\forall x \in H$, $x \neq 0$. If $(Ax, x) - (Bx, x) \ge 0$, $\forall x \in H$, then we write $A \ge B$, etc.

We shall also use the notation

$$y = y_k, \quad \hat{y} = y_{k+1}, \quad \check{y} = y_{k-1}, \quad y_t = (\hat{y} - y)/\tau, \quad y_{\bar{t}} = (y - \check{y})/\tau,$$

$$y_{\overset{\circ}{t}} = (\hat{y} - \check{y})/2\tau, \quad y_{t\bar{t}} = (\hat{y} - 2y + \check{y})/\tau^2, \quad \|y\|_A = \sqrt{(Ay, y)},$$

$$\|y\|_B = \sqrt{(By, y)}, \quad \|y\|_{B^{-1}} = \sqrt{(B^{-1}y, y)}.$$

3. Any three-level scheme can be written in the form

$$By_{t\bar{t}} + B_0 y_{\overset{\circ}{t}} + Ay = \varphi, \quad 0 < t = k\tau < t_0, \quad y(0) = y_0, \quad y_t(0) = \bar{y}_0, \tag{1}$$

where $y_0, \bar{y}_0$ are arbitrary elements of H; $\varphi = \varphi(t)$ is an arbitrary function; $B = B(t)$, $A = A(t)$, $B_0 = B_0(t)$, $t \in \omega_\tau$.

In ([1]), three-level schemes were studied that were written in one of the forms

$$By_{\bar{t}} + \tau^2 R y_{t\bar{t}} + Ay = \varphi, \tag{2_1}$$

$$(E + \tau^2 R) y_{t\bar{t}} + B y_{\overset{\circ}{t}} + Ay = \varphi. \tag{2_2}$$

For convenience of exposition we are forced to change the notation and to write any three-level scheme in the canonical form (1). Comparing (1) with (2_1) and (2_2), we see that

$$B = \tau^2 R, \quad B_0 = B \text{ for } (2_1); \qquad B = E + \tau^2 R, \quad B_0 = B \text{ for } (2_2).$$

Keeping these relations in mind, it is easy to use the results of ([1]) for scheme (1).

The solution of problem (1) is the sum of the solutions of two problems

$$By_{\bar{t}} + B_0 y_{\overset{\circ}{t}} + Ay = 0, \quad y(0) = y_0, \quad y_t(0) = \bar{y}_0; \tag{1a}$$

$$By_{t\bar{t}} + B_0 y_{\overset{\circ}{t}} + Ay = \varphi, \quad y(0) = 0, \quad y_t(0) = 0. \tag{1 }$$

We shall assume that the original family of schemes (1) is specified by requirements on the operators

$$A = A^* > 0, \qquad B = B^* > 0, \qquad B_0 \geqslant 0, \qquad \forall t \in \omega_\tau; \tag{3}$$

$$(1 - c_0\tau)\check{A} \leqslant A \leqslant (1 + c_0\tau)\check{A}, \qquad (1 - c_0\tau)\check{B} \leqslant \hat{B} \leqslant (1 + c_0\tau)\check{B}, \tag{4}$$

where $c_0 = \text{const} > 0$ does not depend on h and τ. Conditions (4) mean that A and B satisfy Lipschitz conditions with respect to t. It follows from (1) that the condition

$$B \geqslant \frac{1+\varepsilon}{4}\tau^2 A \quad \left(R \geqslant \frac{1+\varepsilon}{4} A \quad \text{for } B = \tau^2 R\right), \tag{5}$$

where $\varepsilon = \text{const} > 0$ is an arbitrary constant independent of h and τ, is sufficient for the stability of scheme (1) with respect to the initial data, so that for scheme (1a) the estimate

$$\|Y_{k+1}\|_* \leqslant M_0 \|Y_1\|_*, \tag{6}$$

holds,

$$\|Y_{k+1}\|_*^2 = \frac{1}{4}\|y_{k+1} + y_k\|^2_{A(t_k)} + ((B(t_k) - \frac{1}{4}\tau^2 A(t_k))y_{t,k}, y_{t,k}), \quad k > 0; \tag{7}$$

$$\|Y_1\|_*^2 = \frac{1}{4}\|y_1 + y_0\|^2_{A(\tau)} + ((B(\tau) - \frac{1}{4}\tau^2 A(\tau))y_{t,0}, y_{t,0}), \tag{8}$$

$$\|y\|^2_{A(t_k)} = (A(t_k)y, y),$$

$M_0 = M_0(c_0, \varepsilon) \geqslant 1$, $M_0 = 1$ if A and B are constant (not depending on t) operators, i.e. $c_0 = 0$.

4. Lemma 1. *If conditions (3) and (5) are satisfied, then*

$$\|Y_{k+1}\|_* \leqslant \|y_k\|_{A(t_k)} + \|y_{t,k}\|_{B(t_k)}; \tag{9}$$

$$\|Y_{k+1}\|_*^2 \geqslant \frac{\varepsilon}{1+\varepsilon}\|y_{k+1}\|^2_{A(t_k)}; \tag{10}$$

$$\|Y_{k+1}\|_*^2 \geqslant \beta\big(\|y_{k+1}\|_{A(t_k)} + \|y_{t,k}\|_{B(t_k)}\big)^2, \qquad \beta = \frac{1}{4}\varepsilon/(1+\varepsilon). \tag{11}$$

Proof. Denote

$$\hat{J} = \frac{1}{4}\|\hat{y} + y\|^2_A + ((B - \frac{1}{4}\tau^2 A)y_t, y_t) = (Ay, \hat{y}) + \|y_t\|^2_B.$$

1) $\hat{J} = (Ay, y + \tau y_t) + \|y_t\|^2_B = \|y\|^2_A + \tau(Ay, y_t) + \|y_t\|^2_B \leqslant \|y\|^2_A + \tau\|y\|_A\|y_t\|_A + \|y_t\|^2_B$. By virtue of (5) we have

$$\tau\|y_t\|_A \leqslant \frac{2}{\sqrt{1+\varepsilon}}\|y_t\|_B,$$

$$\hat{J} \leqslant \|y\|^2_A + \frac{2}{\sqrt{1+\varepsilon}}\|y\|_A\|y_t\|_B + \|y_t\|^2_B \leqslant \big(\|y\|_A + \|y_t\|_B\big)^2.$$

Hence (9) follows.

2) $\hat{J} = (A\hat{y}, y) + \|y_t\|_B^2 = (A\hat{y}, \hat{y} - \tau y_t) + \|y_t\|_B^2 =$

$$= \|\hat{y}\|_A^2 - \tau(A\hat{y}, y_t) + \|y_t\|_B^2 \geqslant \|\hat{y}\|_A^2 - \tau\|\hat{y}\|_A\|y_t\|_A + \|y_t\|_B^2 \geqslant$$

$$\geqslant \|\hat{y}\|_A^2 - \frac{2}{\sqrt{1+\varepsilon}}\|\hat{y}\|_A\|y_t\|_B + \|y_t\|_B^2, \tag{12}$$

$$\hat{J} \geqslant (1 - c_0)\|\hat{y}\|_A^2 + \left(1 - \frac{1}{c_0(1+\varepsilon)}\right)\|y_t\|_B^2.$$

Choosing $c_0 = 1/(1+\varepsilon)$, we obtain (10).

3) From (12) follows (11).

Substituting then (9) into (8), and (10) and (11) into (7), and using (6), we see that the following is valid.

Theorem 1. *Let conditions (3)–(5) be satisfied. Then scheme (1) is stable in H_A with respect to the initial data, so that for (1a) the estimates hold:*

$$\|y_{k+1}\|_{A(t_k)} \leqslant M_0\sqrt{\frac{1+\varepsilon}{\varepsilon}}(\|y(0)\|_{A(\tau)} + \|y_t(0)\|_{B(\tau)}), \tag{13}$$

$$\|y_{k+1}\|_{A(t_k)} + \|y_t, k\|_{B(t_k)} \leq M_0\, 2\sqrt{\frac{1+\varepsilon}{\varepsilon}}\,(\|y(0)\|_{A(\tau)} + \|y_t(0)\|_{B(\tau)}). \tag{13}$$

5. From the stability (13) with respect to the initial data of scheme (1) there follows its stability with respect to the right-hand side.

Theorem 2. *If conditions (3)–(5) are satisfied, then for the solution of problem (16) the a priori estimate holds*

$$\|y_{k+1}\|_{A(t_k)} \leq M_0\sqrt{\frac{1+\varepsilon}{\varepsilon}}\sum_{j=1}^{k}\tau\|\varphi_j\|_{B^{-1}(t_j)}. \tag{14}$$

Proof. We seek the solution of problem (16) in the form

$$y_k = \sum_{j=1}^{k}\tau G_{k,j}, \qquad k = 1, 2, \dots, \quad y_0 = 0, \tag{15}$$

where $G_{k,j}$, as a function of k, for any fixed $j = 1, 2, \dots$ satisfies equation (1a) and the initial conditions

$$G_{j,j} = 0, \qquad (0.5\tau B_0(t_j) + B(t_j))(G_{j+1,j} - G_{j,j})/\tau = \varphi_j.$$

Hence, and from (15), it follows that $y(0) = y_t(0) = 0$. Since $B_0 \geq 0$, $B = B^* > 0$, then for the solution of the equation $(0.5\tau B_0 + B)w = \varphi$ we have $\|w\|_B \leq \|\varphi\|_{B^{-1}}$, i.e. $\|(G_t)_{j,j}\|_{B(t_j)} \leq \|\varphi_j\|_{B^{-1}(t_j)}$. By Theorem 1,

$$\|G_{k+1,j}\|_{A(t_k)} \leq M_0\sqrt{(1+\varepsilon)/\varepsilon}\,\|(G_t)_{j,j}\|_{B^{-1}(t_j)}.$$

Using then the triangle inequality and (13),

$$\|y_{k+1}\|_{A(t_k)} \leq \sum_{j=1}^{k} \tau\|G_{k+1,j}\|_{A(t_k)},$$

we obtain (14).

6. Let us now consider the family of schemes

$$By_{\bar{t}t} + Ay = \varphi(t), \qquad 0 < t = k\tau < t_0, \qquad y(0) = y_0, \qquad y_t(0) = \bar{y}_0. \quad (16)$$

We shall, as usual, denote by (16a) problem (16) with $\varphi = 0$, and by (16b) problem (16) with homogeneous initial data $y(0) = 0$, $y_t(0) = 0$. Suppose that

$$A \text{ and } B \text{ are constant operators}, \quad A = A^* > 0,\ B = B^* > 0, \quad (17)$$

and condition (5) is valid.

Then scheme (16) is equivalent (cf. (2)) to the explicit scheme

$$x_{\bar{t}t} + Cx = \tilde{\varphi}, \qquad 0 < t = \omega\tau, \qquad x(0) = x_0, \qquad x_t(0) = \bar{x}_0, \quad (18)$$

where $C = B^{-1/2}AB^{-1/2}$, $x = B^{1/2}y$, $\tilde{\varphi} = B^{-1/2}\varphi$. Let us rewrite it in the form

$$C^{-1}x_{\bar{t}t} + x = \Phi, \qquad \Phi = C^{-1}\tilde{\varphi} = B^{1/2}A^{-1}\varphi, \quad (18^*)$$

and apply to it Theorems 1 and 2 (with $A = E$, $B_0 = 0$, $B = C^{-1}$),

$$\|x_{k+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left[\|x_0\| + \|x_{t,0}\|_{C^{-1}} + \sum_{j=1}^{k}\tau\|\Phi_j\|_C\right],$$

taking into account that $M_0 = 1$. Since

$\|x_k\| = \|y_k\|_B, \qquad \|\Phi\|_C^2 = (C\Phi,\Phi) = (CC^{-1}\tilde{\varphi}, C^{-1}\tilde{\varphi}) = (C^{-1}\tilde{\varphi},\tilde{\varphi}) = (B^{1/2}A^{-1}\varphi, B^{-1/2}\varphi) = (A^{-1}\varphi,\varphi),$

$\|\Phi\|_C = \|\varphi\|_{A^{-1}}$, as a result, for (16) we obtain the estimate

$$\|y_{k+1}\|_B \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left[\|y(0)\|_B + \|By_t(0)\|_{A^{-1}} + \sum_{j=1}^{k} \tau \|\varphi_j\|_{A^{-1}} \right]. \tag{19}$$

Theorem 3. *Let (5) and (17) be satisfied. Then for scheme (16) the a priori estimate (19) is valid.*

Corollary. For an explicit scheme ($B = E$), under the condition $E \geq \frac{1}{4}(1+\varepsilon)\tau^2 A$ ($\varepsilon > 0$ arbitrary), the estimate holds

$$\|y_{k+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left[\|y(0)\| + \|y_t(0)\|_{A^{-1}} + \sum_{j=1}^{k} \tau \|\varphi_j\|_{A^{-1}} \right]. \tag{20}$$

Remark 1. It follows from (20) that the second initial value is taken in a weaker norm. Estimate (20) is convenient for investigating the rate of convergence in the grid norm L_2 of homogeneous difference schemes for equations with discontinuous coefficients.

Remark 2. If

$$B = E + \tau^2 R, \qquad A \text{ and } R \text{ are constant operators}, \quad R \geq 0, \tag{21}$$

then from (5), for problem (16), there follows the estimate

$$\|y_{k+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{j=1}^{k} \tau \|\varphi_j\|_{A^{-1}} \qquad \text{for } B \geq \frac{1+\varepsilon}{4}\tau^2 A.$$

If, instead of (5), condition (see (1)) $R \geq \frac{1}{4}A$ is satisfied, then for (16) we obtain

$$\|y_{k+1}\| \leq \sqrt{2}\, t_{k+1} \sum_{j=1}^{k} \tau \|\varphi_j\|. \tag{22}$$

7. Estimate (20) can be obtained by the method of separation of variables in the case of finite-dimensional H and constant A.

For a variable operator $A = A(t)$, the method of separation of variables is ineffective.

If $A = A(t)$ is variable and B is a constant operator, then an estimate of the form (19) holds if

$$\rho^{-1} A \leq \hat{A} \leq \rho A, \qquad \rho = e^{c_0 \tau} \quad \text{for all } t \in \omega_\tau, \tag{23}$$

where $c_0 = \text{const} > 0$ does not depend on τ and h. In this case, in (19) the right-hand side is multiplied by a constant $M = M(c_0) > 1$. Comparing (6) with (23), we see that (23) follows from (6) for sufficiently small $\tau < \tau_0(c_0)$. In the case of scheme (1) of general form, Theorem 3 remains valid if $B_0 = B_0^*$ and the operators A, B, B_0 are permutable, while A and B are constant.

8. Let us consider, as an example, scheme (16) with weights (see (1)). For it,

$$B = E + \sigma\tau^2 A.$$

Condition (5) is satisfied for

$$\sigma \geq (1+\varepsilon)/4 - 1/(\tau^2\|A\|).$$

In particular, the explicit scheme ($\sigma = 0$) is stable if

$$\tau \leq \frac{2}{\sqrt{1+\varepsilon}} \frac{1}{\sqrt{\|A\|}}.$$

Thus, for the equation of vibrations of a string,

$$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2, \quad 0 < x < 1, \quad t > 0, \quad u(0,t) = u(1,t) = 0,$$

$$u(x,0) = u_0(x), \qquad \partial u(x,0)/\partial t = \bar{u}_0(x),$$

we have $A = -\Lambda$, $\Lambda y = y_{\bar{x}x}$ (see (4)), and the explicit scheme is stable for

$$\tau \leq h/\sqrt{1+\varepsilon},$$

where h is the grid step on the interval $0 \leq x \leq 1$.

Theorems 1, 2, 3 are applicable to the investigation and construction, by the regularization method (1), of difference schemes (in particular, economical schemes) for equations and systems of equations of mathematical physics.

Institute of Applied Mathematics
Academy of Sciences of the USSR
Moscow

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REFERENCES

1. A. A. Samarskii, *Zhurn. vychislit. matem. i matem. fiz.*, **7**, No. 1, 62 (1967); **7**, No. 5, 1093 (1967).

2. A. A. Samarskii, *DAN*, **181**, No. 4, 808 (1968).

3. A. V. Gulin, *Zhurn. vychislit. matem. i matem. fiz.*, **8**, No. 4, 899 (1968).

4. A. A. Samarskii, *Lectures on the Theory of Difference Schemes*, Ch. IV, Moscow, 1969.

Note: Figure translations are in progress. See original paper for figures.