ON ONE CLASS OF INTEGRAL EQUATIONS OF CONVOLUTION TYPE AND ITS APPLICATION

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ON ONE CLASS OF INTEGRAL EQUATIONS OF CONVOLUTION TYPE AND ITS APPLICATION

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The following generalization of integral equations of convolution type is considered:

$$H\varphi \equiv a_0(t)\varphi(t) + \sum_{j=1}^{n} a_j(t) \int_{-\infty}^{\infty} b_j(\tau) h_j(t-\tau)\varphi(\tau)\, d\tau = f(t), \qquad -\infty < t < \infty. \tag{1}$$

Here and everywhere below $h_j(t) \in \mathcal{L}_1(-\infty,\infty)$; $a_j(t)$ and $b_j(t)$ are bounded measurable functions, with $|a_0(t)| \geq c > 0$; $\varphi(t), f(t) \in E^*$.

Various special cases of equation (1) (mainly cases of piecewise-constant coefficients) have been studied, for example, in ([2–6]). The general equation of the form (1) with $b_j(t) \equiv 1$, $j = 1, 2, \dots, n$, was investigated by L. S. Rakovshchik ([1]) under the assumption that $\varphi(t), f(t) \in \mathcal{L}_p(-\infty,\infty)$, $p > 1$.

In this note the results of ([1]) are generalized to the operator H in the case of any of the spaces E. The results obtained are applied to the study of one class of integral equations of the form

$$K\psi \equiv \psi(x) + \sum_{j=1}^{n} c_j(x) \int_0^a d_j(y) k_j(x,y)\psi(y)\, dy = g(x), \qquad 0 < x < a, \tag{2}$$

where $k_j(x,y)$ are homogeneous functions of arbitrary orders α_j; $c_j(x)$, $d_j(x)$ belong to certain subclasses of bounded measurable functions. Necessary and sufficient conditions will be found for the Noether property of the operator K in a series of weighted spaces E_β, and the index will be computed. We note that equation (2) with $\alpha_j = -1$, $c_j(x), d_j(x) \in C(0,a)$ belongs to the class of equations studied by L. G. Mikhailov in ([12]).

No. 1. Let us introduce the spaces M_0^{sup}, M_0^{mes}, M^{sup}, M^{mes} of bounded measurable functions:

$$M^{\text{sup}} = M_0^{\text{sup}} \oplus X, \qquad M^{\text{mes}} = M_0^{\text{mes}} \oplus X,$$

$$M_0^{\text{sup}} = \{\varphi(t) \mid \varphi(t) \in M;\ \lim_{n\to\infty} \sup_{|t|>n} |\varphi(t)| = 0\},$$

$$M_0^{\text{mes}} = \{\varphi(t) \mid \varphi(t) \in M;\ \lim_{n\to\infty} \text{mes}(t : |\varphi(t)| > \varepsilon,\ |t| > n;\ \varepsilon > 0) = 0\},$$

where X is the two-dimensional space with basis $\theta(x)$, $\theta(-x)$, $\theta(x) = \frac{1}{2}(1 + \operatorname{sign} x)$. Obviously, $\varphi(t) \in M^{\text{sup}}$ (or M^{mes}) if and only if there exist constants $c_\pm$ such that $\theta(\pm t)[\varphi(t) - c_\pm] \in M_0^{\text{sup}}$ (or M_0^{mes}, respectively). The constants $c_\pm$ will be called the (generalized) values of the bounded measurable function at infinity: $c_\pm = \varphi(\pm\infty)$ (cf. ([1])).

*

We note that to the series E of Banach spaces $\mathcal{L}_p, M, M^u, M^c, C, C^0$ ([7]) we add the subspaces important for us, M^{sup}, M^{mes}, of bounded measurable functions having (in a certain sense) values at infinity.

We note the following properties of the classes $M_0^{\text{sup}}, M_0^{\text{mes}}, M^{\text{sup}}, M^{\text{mes}}$:
1) they are complete in the norm M; 2) functions from M^{mes} are continuous in measure on the whole axis; 3) $\varphi(t) \in M_0^{\text{sup}}$ if and only if $\varphi(t)$ has a majorant from C^0: $|\varphi(t)| \le a(t) \in C^0$; 4) functions of bounded variation (on the closed axis) belong to M^{sup}; 5) $M \cap \mathcal{L}_p \subset M_0^{\text{mes}}$ for any $p \ge 1$.

Everywhere below we shall assume that $E = \mathcal{L}_1,\ \mathcal{L}_p,\ M,\ M^{\text{sup}},\ M^{\text{mes}},\ M_0^{\text{sup}},\ M_0^{\text{mes}},\ M^u,\ M^c,\ C,\ C^0$, where the spaces $E = M^u,\ M^c,\ C,\ C^0$ of continuous functions will be understood in the following sense: $E = E_+ \oplus E_-$ (see (8)).

Lemma 1. *If $h(t) \in \mathcal{L}_1$, then the convolution operator $A\varphi = h * \varphi$ acts from M_0^{mes} into C and from M_0^{mes} into C^0.*

No. 2. Put, without loss of generality, $a_0(t) \equiv 1$. Let $a_j(t), b_j(t) \in M^{\text{mes}}$. Represent the operator H in the form

$$H\varphi = \Pi\varphi + T_1\varphi + T_2\varphi, \tag{3}$$

where

$$\Pi\varphi = \begin{cases} \varphi(t) + \int_{-\infty}^{\infty} \sum_{j=1}^{n} a_j(\infty)b_j(\infty)h_j(t-\tau)\varphi(\tau)\,d\tau, & t > 0, \\ \varphi(t) + \int_{-\infty}^{\infty} \sum_{j=1}^{n} a_j(-\infty)b_j(-\infty)h_j(t-\tau)\varphi(\tau)\,d\tau, & t < 0. \end{cases}$$

is a paired operator, and

$$T_1\varphi = \sum_{j=1}^{n} \tilde{a}_j(h_j * b_j\varphi) + \sum_{j=1}^{n} (a_j - \tilde{a}_j)(h_j * \tilde{b}_j\varphi),$$

$$\tilde{a}_j(t) = a_j(t) - a_j(\infty)\theta(t) - a_j(-\infty)\theta(-t); \qquad \tilde{b}_j(t) = b_j(t) - b_j(\infty)\theta(t) -$$

$$- b_j(-\infty)\theta(-t),$$

$$T_2\varphi = \sum_{j=1}^{n} (b_j(\infty) - b_j(-\infty))a_j(-\infty)\theta(-t) \int_{-\infty}^{\infty} \theta(\tau)h_j(t-\tau)\varphi(\tau)\,d\tau -$$

$$- \sum_{j=1}^{n} (b_j(\infty) - b_j(-\infty))a_j(\infty)\theta(t) \int_{-\infty}^{\infty} \theta(-\tau)h_j(t-\tau)\varphi(\tau)\,d\tau.$$

Operators of the form T_2 are completely continuous ([9,10]) in all spaces E. For a broad class of coefficients this is also true for the operator T_1. Namely, the following holds.

Lemma 2. *The operator $T\varphi = a(h * b\varphi)$, $h(t) \in \mathcal{L}_1$, $a(t), b(t) \in M$, is completely continuous in the space E, if:*

1) *either*

$$a(t) \in \begin{cases} M_0^{\text{mes}} & \text{for } E = \mathcal{L}_p \ (p \geq 1), \\ M_0^{\text{sup}} & \text{for } E = M,\ M_0^{\text{sup}},\ M_0^{\text{mes}},\ M^{\text{sup}},\ M^{\text{mes}}, \\ C_0 & \text{for } E = M^u,\ M^c,\ C,\ C^0; \end{cases} \tag{4}$$

2) *or*

$$b(t) \in \begin{cases} M_0^{\text{sup}} & \text{for } E = \mathcal{L}_1, \\ M_0^{\text{mes}} & \text{for the remaining } E; \end{cases} \tag{5}$$

$$a(t) \in C, \quad \text{if } E = M^u,\ M^c,\ C,\ C^0.$$

For $b(t) \equiv 1$, $a(t) \in M_0^{\mathrm{mes}}$, $E = \mathcal{L}_p$ $(p > 1)$, the assertion of Lemma 2 was obtained in ([1]).

By virtue of the complete continuity of the operators T_1, T_2 from (3), we obtain the following theorem:

Theorem 1. *Let $h_j(t) \in \mathcal{L}_1$, $j = 1, 2, \dots, n$, and let $a_j(t)$, $j = 0, 1, \dots, n$, and $b_j(t)$, $j = 1, 2, \dots, n$, satisfy respectively the condi-*

problems (4), (5), in which the index 0 is omitted*. Then the necessary and sufficient condition for the Noether property of the operator H in E has the form

$$\sigma(\lambda)_\pm = a_0(\pm\infty) + \sum_{j=1}^{n} a_j(\pm\infty) b_j(\pm\infty) \mathcal{H}_j(\lambda) \neq 0, \qquad -\infty \leq \lambda \leq \infty,$$

where

$$\mathcal{H}_j(\lambda) = \int_{-\infty}^{\infty} h_j(t) e^{it\lambda}\, dt.$$

Under these conditions

$$\varkappa_E(H) = -\frac{1}{2\pi} \Delta \left[\arg \frac{\sigma(\lambda)^+}{\sigma(\lambda)^-} \right]_{-\infty}^{\infty}.$$

Using one result from the theory of linear operators (([5], Lemma 1), we arrive at the following theorem:

Theorem 2. The operator H has in all spaces E the same zeros and the same solvability conditions**, if one of the following conditions is fulfilled:

1) $a_j(t) \in M^{\mathrm{mes}}$, $\quad b_j(t) \in M^{\mathrm{mes}}$, $\quad E = \mathcal{L}_p(p > 1)$;
2) $a_j(t) \in M^{\mathrm{mes}}$, $\quad b_j(t) \in M^{\mathrm{sup}}$, $\quad E = \mathcal{L}_p(p \geq 1)$;
3) $a_j(t) \in M^{\mathrm{sup}}$, $\quad b_j(t) \in M^{\mathrm{sup}}$, $\quad E = \mathcal{L}_p(p \geq 1)$, $\quad M$, M_0^{mes}, M_0^{sup}, M^{mes}, M^{sup};
4) $a_j(H \in M^{\mathrm{sup}}$, $\quad b_j(t) \in M^{\mathrm{mes}}$, $\quad E = \mathcal{L}_p(p > 1)$, $\quad M$, M_0^{mes}, M_0^{sup}, M^{mes}, M^{sup};
5) $a_j(t) \in C$, $\quad b_j(t) M^{\mathrm{mes}}$; $\quad E$—arbitrary, except, perhaps, $\mathcal{L}_1$;
6) $a_j(t) \in C$, $\quad b_j(t) \in M^{\mathrm{sup}}$, $\quad E$—arbitrary.

No. 3. In equation (2) we shall assume (cf. ([11])) that there exists a number β such that

$$\int_0^\infty |k_j(1,y)|\, y^{-\beta}\, dy < \infty, \qquad j = 1, 2, \dots, n. \tag{6}$$

Using a well-known device (([13], p. 396), we reduce equation (2), by means of the transformation $x = ae^{-t}$, $y = ae^{-\tau}$, $\varphi(t) = e^{-\beta t}\psi(x)$, $f(t) = e^{-\beta t} g(x)$, to an equation on the semiaxis of the form (1). In this case

$$h_j(t) = a^{1+\alpha_j} e^{(1-\beta)t} k_j(1, e^t) \in \mathcal{L}_1,$$

$$a_j(t) = \theta(t) c_j(ae^{-t}) e^{-(1+\alpha_j)t}, \qquad b_j(t) = \theta(t) d_j(ae^{-t}).$$

The indicated transformation establishes an isometry between the space $E(0,\infty)$ and the weighted space $E_\beta = E_\beta(0,a) = \{\psi \mid e^{-\beta t}\psi(ae^{-t}) = \varphi(t) \in E(0,\infty)\}$, $\|\psi\|_{E_\beta} = \|\varphi\|_{E_+}$. Obviously,

$$E_\beta = \begin{cases} x^{-\beta} E(0,a), & E \neq \mathcal{L}_p, \\ x^{-\beta+1/p} E(0,a), & E = \mathcal{L}_p, \end{cases}$$

and if U is the operator realizing the isometry of E_β onto E_+, then $K = U^{-1} H U$.

We shall assume that in (2) $\psi(x), g(x) \in E_\beta$. Applying the results obtained for the operator H, we arrive at the following theorem:

Theorem 3. Let the kernels $k_j(x,y)$ be homogeneous functions of orders α_j, $-\infty < \alpha_j < \infty$, $j = 1, 2, \dots, n$, satisfying condition (6), and suppose also that $d_j(x) \in C(0,a)$, $x^{1+\alpha_j} c_j(x) \in C(0,a)$. Then the necessary and sufficient condition for the Noether property of the operator K in E_β has the form

$$\sigma(\lambda) = 1 + \sum_{j=1}^{n} a^{1+\alpha_j} d_j(0) \lim_{x\to 0} \left[x^{1+\alpha_j} c_j(x)\right] K_j(i\lambda - \beta + 1) \neq 0,$$

$$-\infty \leq \lambda \leq \infty,$$

* That is, the functions $\tilde{a}_j(t), \tilde{b}_j(t)$ satisfy the very same conditions (4), (5).

** The solvability conditions may be interpreted as orthogonality conditions to the zeros of the transposed operator, for example, in $\mathcal{L}_p$ $(p > 1)$.

where

$$K_j(s) = \int_0^\infty k_j(1,x) x^{s-1}\, dx$$

is the Mellin transform of the function $k_j(1,x)$. If this condition is satisfied,

$$\varkappa_{E_\beta}(K) = -\frac{1}{2\pi}\Delta[\arg\sigma(\lambda)]_{-\infty}^{\infty}.$$

Let us note that the formulation of Theorem 3, for simplicity, is given for the case of continuous coefficients $x^{1+\alpha_j}c_j(x)$, $d_j(x)$. It is not difficult to obtain a theorem for more general coefficients, for example, for $c_j(x)x^{1+\alpha_j}$, $d_j(x) \in M^{\sup}(0,a)$, where

$$M^{\sup}(0,a) = \{\psi(x) \mid \psi(x) \in M(0,a);\ \lim_{n\to\infty} \sup_{0<x<1/n} |\psi(x) - c| = 0\}, \quad c = \text{const}.$$

Similarly to Theorem 3, from Theorem 2 there follows a theorem on the zeros of the operator K, on whose formulation we shall not dwell. We also note that analogous theorems hold in the case $a = \infty$.

As an example, consider the equation

$$K_1\psi \equiv \psi(x) + \int_0^a \sum_{j=1}^n k_j(x,y)\psi(y)\,dy = f(x), \qquad 0 < x < a, \tag{7}$$

where $k_j(\lambda x, \lambda y) = \lambda^{\alpha_j}k_j(x,y)$. In the case $n = 1$ and $\alpha_1 = -1$ this equation was studied by L. G. Mikhailov ([11]). Applying Theorem 3, we see that, for example, when $\alpha_j > -1$, $j = 1,2,\dots,n$, equation (7) is Fredholm in all spaces E_β: $\varkappa_{E_\beta}(K_1) = 0$ provided condition (6) is satisfied. It can be shown, using a theorem of L. Hörmander ([14]), p. 9, that the operator (7) is unbounded in any of the spaces E_β if $\alpha_j < -1$ at least for one j, $j = 1,2,\dots,n$. And, finally, we note that when $n = 1$, $\alpha_j = \alpha \geq -1$, the zeros of the operator (7) have the form $\psi(x) = x^{1+\alpha-\beta}\psi_0(x)$, where $\psi_0(x)$ is continuous and $\psi(0) = 0$.

As a second example, consider the equation

$$K_2\psi \equiv \psi(x) + c(x)\int_0^1 d(y)\frac{\ln y - \ln x}{y - x}\psi(y)\,dy = g(x), \qquad 0 < x < 1,$$

in the spaces $\mathcal{L}_p$ $(1 < p < \infty)$. Let $c(x), d(x) \in C(0,1)$ and

$$v_p = \frac{1}{\pi}\sin\frac{\pi}{p}.$$

The necessary and sufficient condition for Noetherianity has the form

$$c(0)d(0) > -\frac{1}{\pi^2} = -v_2^2$$

for $p = 2$, and

$$c(0)d(0) \neq -v_p^2$$

for $p \neq 2$. In this case $\varkappa_{\mathcal{L}_p}(K_2) = 0$ if $c(0)d(0) > -v_p^2,\ 1 < p < \infty$, and

$$\varkappa_{\mathcal{L}_p}(K_2) = \operatorname{sign}(2 - p)$$

if $c(0)d(0) < -v_p^2,\ p \neq 2$.

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References

1. L. S. Rakovshchik, UMN, 18, no. 4, 171 (1963).
2. L. S. Rakovshchik, Sibirsk. matem. zhurn., 6, no. 1, 186 (1965).
3. I. B. Simonenko, Izv. vyssh. uchebn. zaved., Matematika, no. 2 (9), 213 (1959).
4. F. D. Berkovich, ibid., no. 12, 15 (1967).
5. I. A. Feldman, Izv. AN MSSR, no. 10 (188), 16 (1961).
6. I. I. Komyak, DAN, 179, no. 2, 279 (1968).
7. M. G. Krein, UMN, 13, no. 5 (8), 3 (1958).
8. I. Ts. Gokhberg, M. G. Krein, Teoret. i prikl. matem., vol. 1, 58, L' vov (1959).
9. I. Ts. Gokhberg, M. G. Krein, UMN, 13, no. 2 (80), 3 (1958).
10. I. B. Simonenko, Matem. sborn., 174 (116), no. 2, 298 (1967).
11. L. G. Mikhailov, Integral equations with a kernel of one-dimensional degree–1, Dushanbe, 1966.

12. L. G. Mikhailov, Differential and integral equations with singular coefficients, Dushanbe, 1969.

13. E. Titchmarsh, Introduction to the Theory of Fourier Integrals, Moscow –Leningrad, 1948.

14. L. Hörmander, Estimates for operators invariant with respect to translation, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.