CONNECTED COMPONENTS OF NORMALLY SOLVABLE ELLIPTIC SYSTEMS IN THE PLANE

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Abstract

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CONNECTED COMPONENTS OF NORMALLY SOLVABLE ELLIPTIC SYSTEMS IN THE PLANE

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Let D be a simply connected two-dimensional domain bounded by an n-times smooth Lyapunov curve Γ. Consider the boundary-value problem for an elliptic system of equations

$$\mathcal{L}u = \sum_{s=0}^{n} A_s(z)\frac{\partial^n u}{\partial x^s \partial y^{n-s}} + \sum_{0\le k+l\le n-1} A_{k,l}(z)\frac{\partial^{k+l}u}{\partial x^k \partial y^l} = f(z), \qquad z\in \overline{D}, \quad (1)$$

$$\Lambda u = \sum_{0\le k+l\le n-1} B_{k,l}(z)\frac{\partial^{k+l}u}{\partial x^k \partial y^l}\bigg|_{\Gamma} = \psi(z), \qquad z=(x,y). \quad (1')$$

In formula (1), $A_s(z)$, $A_{k,l}(z)$ are real square matrices of order $p \ge 2$; $B_{k,l}(z)$ are rectangular real $r\times p$ matrices; $\psi(z)$ is an r-dimensional column vector; $u(z)$ is the unknown p-dimensional column vector; $r = \frac{1}{2}pn$.

It is assumed that the matrices $A_s(z)$ are twice, and $A_{k,l}(z)$ once, continuously differentiable in some domain $G \supset \overline{D}$. The matrices $B_{k,l}(z)$ are assumed to satisfy a Hölder condition on Γ.

Ellipticity of the system (1) is understood in the sense of I. G. Petrovskii:

$$P(z,\lambda) = \det\left(\sum_{s=0}^{n} A_s(z)\lambda^{n-s}\right) \ne 0, \qquad \operatorname{Im}\lambda = 0, \qquad z\in\overline{D}, \quad (2)$$

$$\det A_0(z) \ne 0, \qquad z\in\overline{D}. \quad (2')$$

It is assumed that the problem (1), (1′) is normally solvable. Referring the reader, concerning this notion, to the fundamental works ([1],[2]), we introduce into consideration the $2r\times 2r$ matrix

$$\mathfrak{A}(z) = \left\| \begin{array}{ccccc} 0 & E_p & 0 & \cdots & 0 \\ 0 & 0 & E_p & \cdots & 0 \\ \cdots & \cdots & \bullet & \cdots & \cdots \\ 0 & 0 & 0 & & 0 \\ -A_0^{-1}A_n & -A_0^{-1}A_{n-1} & -A_0^{-1}A_{n-2} & \cdots & -A_0^{-1}A_1 \end{array} \right\|.$$

The spectrum of the matrix $\mathfrak{A}(z)$ coincides with the roots of the polynomial (2) for each $z \in \overline{D}$. Consider the invariant subspace of the matrix $\mathfrak{A}(z)$ corresponding to the spectrum from the upper half-plane. Choose in this subspace a basis:

$$F(z) = \left\| \begin{array}{c} F_1(z) \\ F_2(z) \end{array} \right\|, \tag{3}$$

where $F_1(z)$ and $F_2(z)$ are $r \times r$ matrices, which may be chosen continuously differentiable with respect to x and y (cf. ([2],[3])).

Denote also by $\|B_1(z)B_2(z)\|$, $z \in \Gamma$, the $r \times 2r$ matrix formed by the columns of the matrices $B_{n-1,0}(z), B_{n-2,0}(z), \dots, B_0$.

Let

$$B_1(z)F_1(z) + B_2(z)F_2(z) = \Omega(z). \tag{4}$$

The necessary and sufficient condition for normal solvability is that

$$\det\Omega(z) \neq 0, \qquad z \in \Gamma \tag{5}$$

(see ([2,4])). In this case the index $\varkappa$ of problem (1), (1′) turns out to be equal to

$$\varkappa = -\frac{1}{\pi}\operatorname{Arg}\det\Omega(z)\Big|_{\Gamma} + rn \tag{6}$$

(see ([2]), p. 75).

Definition 1. Two elliptic normally solvable systems of the form (1), (1′): $\{\mathcal{L}^{(1)}, \Lambda^{(1)}\}$ and $\{\mathcal{L}^{(2)}, \Lambda^{(2)}\}$, are called **topologically equivalent** (**homotopic**) if there exists a continuous real deformation

$$A_s(\tau, z), \qquad 0 \le \tau \le 1, \quad z \in \bar{D}, \quad s = 0, 1, 2, \dots, n, \tag{7}$$

$$B_{k,l}(\tau, z), \qquad 0 \le \tau \le 1, \quad z \in \Gamma, \quad k + l = n - 1, \tag{7'}$$

taking the coefficient matrices $A_s^{(1)}(z)$ into $A_s^{(2)}(z)$ and $B_{k,l}^{(1)}(z)$ into $B_{k,l}^{(2)}(z)$, respectively, and preserving the ellipticity condition (2) and the condition of normal solvability (5). It is assumed that the deformation (7) preserves the dimensions of the matrices and smoothness.

Let us pose the question of the number of connected components of the set $\{\mathcal{L}, \Lambda\}$ of normally solvable systems. In the case of constant matrices A_s and $B_{k,l}$, $k+l=n-1$, this problem was solved in a paper ([5]) by one of the authors (see also ([6])). In this case the set $\{\mathcal{L}, \Lambda\}$ $(p>2)$ splits into four connected components*. A complete system of invariants has the form

$$\delta_1 = \operatorname{sign}\left(\det \sum_{s=0}^{n} A_s \lambda^{n-s}\right), \qquad \operatorname{Im}\lambda = 0, \tag{8}$$

$$\delta_2 = \operatorname{sign} i^r \det \begin{Vmatrix} F_1 & \bar{F}_1 \\ F_2 & \bar{F}_2 \end{Vmatrix}. \tag{8'}$$

We note that the invariants (8), (8′) do not depend on the matrices $B_{k,l}$. It is easy to verify that the index of problem (1), (1′) in this case is always one and the same:

$$\varkappa = np$$

(see ([6])). The picture naturally changes if the matrices $A_s(z)$ and $B_{k,l}(z)$ depend on z. We formulate the main result of the present article.

Theorem. In order that two systems $\{\mathcal{L}^{(1)}, \Lambda^{(1)}\}$ and $\{\mathcal{L}^{(2)}, \Lambda^{(2)}\}$ be topologically equivalent (see Definition 1), it is necessary and sufficient that the invariants of these systems, computed by formulas (8), (8′), coincide, and, moreover, that the indices of the corresponding problems coincide,

$$\varkappa^{(1)} = \varkappa^{(2)}. \tag{9}$$

The proof makes substantial use of the result of paper ([5]). Along with the equality

$$B_1(z)F_1(z) + B_2(z)F_2(z) = \Omega(z), \tag{10}$$

consider the equality

$$B_1(z)\bar{F}_1(z) + B_2(z)\bar{F}_2(z) = \bar{\Omega}(z). \tag{10'}$$

By virtue of the nonsingularity of the matrix

$$\mathfrak{F}(z) = \begin{Vmatrix} F_1(z) & \bar{F}_1(z) \\ F_2(z) & \bar{F}_2(z) \end{Vmatrix} \tag{11}$$

* The case $p = 2$ was considered in paper ([7]). For the formulation of the problem, see ([8,9]).

for $z \in \overline{D}$ and, consequently, for $z \in \Gamma$ the system (10), (10′), for an arbitrary complex $r \times r$-matrix $\Omega(z)$, $z \in \Gamma$, determines a unique $r \times 2r$-matrix

$$\|B_1(z) B_2(z)\|, \qquad z \in \Gamma, \tag{12}$$

which, as is easy to see, is real. Let us also note that under a continuous deformation of the right-hand sides of the system (10) and (10′) and of the matrix (11), preserving its nonsingularity, the matrix (12) is deformed continuously.

Assuming the domain $\overline{D}$ to be star-shaped* with pole at the origin, let us consider, along with the system of differential equations (1), the system with matrices

$$A_s(t, z) = A_s(tz), \qquad 0 \leq t \leq 1, \quad s = 0, 1, 2, \dots, n. \tag{13}$$

The smooth basis (3) in $\overline{D}$ corresponding to this system has the form

$$F(tz) = \begin{Vmatrix} F_1(tz) \\ F_2(tz) \end{Vmatrix}. \tag{14}$$

Fix in (10) the matrix $\Omega(z)$, $z \in \Gamma$.

Carrying out the deformation (13), we contract the set of matrices $A_s(z)$ to the set of constant matrices $A_s(0)$, $s = 0, 1, 2, \dots, n$. The matrix (11), by virtue of (14), is contracted to the constant matrix $\mathcal{F}(0)$. Obviously, the matrix (12) is thereby deformed to the form $\|\Omega(z)\overline{\Omega}(z)\|\mathcal{F}^{-1}(0)$. The conditions of ellipticity (2) and normal solvability (5) are preserved in the process of deformation.

Thus our problem is reduced to the problem in which the matrices

$$A_s^{(j)} \qquad (s = 0, 1, 2, \dots, n;\ j = 1, 2) \tag{15}$$

are constant. The sets (15) with $j = 1$ and $j = 2$ are homotopic if and only if the invariants (8), (8′) coincide.

It remains to note that the homotopy (7′), by virtue of (10), (10′), is possible if and only if

$$\operatorname{Arg}\det\Omega^{(1)}(z)\big|_{\Gamma} = \operatorname{Arg}\det\Omega^{(2)}(z)\big|_{\Gamma}, \tag{16}$$

which, by virtue of (6), is equivalent to the equality (9). The theorem is proved.

In conclusion we give the canonical form of the system $\{\mathcal{L}, \Lambda\}$ with prescribed δ_1, δ_2, and $\varkappa$. For simplicity we restrict ourselves to the case $n = 2$.

Let first**

$$\delta_1 = 1; \qquad \delta_2 = 1; \qquad \varkappa = 2m. \tag{17}$$

Then

$$\mathcal{L}u = (\partial^2/\partial x^2 + \partial^2/\partial y^2)E_{pu}, \tag{18}$$

$$\Lambda u = \begin{pmatrix} \cos N\varphi & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} \sin N\varphi & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \frac{\partial u}{\partial y}, \tag{19}$$

$$\varphi = \operatorname{Arg} z, \qquad z \in \Gamma, \qquad N = p - m.$$

* Without loss of generality one may take $\overline{D}$ to be a disk.
** In the case $n = 2$, $\varkappa$ is always even (see (6)).

If, however, $\delta_1 = 1,\ \delta_2 = -1,\ \chi = 2m$, then

$$\mathcal{L}u = \begin{pmatrix} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_{p-2} & & 0 \\ & -\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial y^2} & 3\frac{\partial^2}{\partial x\,\partial y} \\ 0 & -3\frac{\partial^2}{\partial x\,\partial y} & \frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial y^2} \end{pmatrix} u, \tag{20}$$

$$\Lambda u = \begin{pmatrix} \cos N\varphi & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 1 \quad 0 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} \sin N\varphi & & 0 \\ & 0 & \\ & & \ddots \\ 0 & & 0_1 \end{pmatrix} \frac{\partial u}{\partial y}. \tag{21}$$

The case $\delta_1 = -1$ is obtained by multiplying $\mathcal{L}$ on the left by $\operatorname{diag}(1, 1, \dots, 1, -1)$.

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Note: Figure translations are in progress. See original paper for figures.