

**(B) -SPACES
GENERATED BY
CONVEX INTEGRANDS
AND
MULTIDIMENSIONAL
VARIATIONAL
PROBLEMS**

MATHEMATICS

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Abstract

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MATHEMATICS

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B-SPACES GENERATED BY CONVEX INTEGRANDS AND MULTIDIMENSIONAL VARIATIONAL PROBLEMS

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This note studies the duality theory for convex multidimensional variational problems (i.e., problems with partial derivatives). In Section 1 functional spaces are described in which it is convenient to consider such problems. These spaces are closely connected with Orlicz spaces ⁽¹⁾ and possess a number of properties inherent in the latter. Section 2 contains the actual formulation of the problem, which is then interpreted in the usual way as a problem on a “shifted” linear manifold. Finally, in Section 3 a duality theorem is formulated for problems of the latter type. This theorem (as applied to the problem considered in Sec. 2) generalizes some results obtained in ^(2, 3) for one-dimensional problems to problems with partial derivatives.

1. Let (T, Σ, μ) be a space with a finite positive measure, $\bar{R} = R \cup \{+\infty\}$. We shall consider functions $f : T \times R^n \rightarrow \bar{R}$ having the following properties:
 - a) $f(t, x)$ is measurable with respect to the σ -algebra $\Sigma \otimes \mathfrak{A}$, where \mathfrak{A} is the algebra of Borel subsets of R^n ,
 - b) for almost every $t \in T$, $f(t, x)$ is convex and closed (i.e., lower semicontinuous) in x .

Such functions were considered in ^(4, 5). They are called in ⁽⁴⁾ normal convex integrands (n.c.i.). For the definition adopted here, see also ⁽⁶⁾. If f is an n.c.i., then $f(t, x(t))$ is measurable for every measurable $x(t)$, and

$$f^*(t, y) = \sup_x ((x, y) - f(t, x))$$

$((\cdot, \cdot)$ is the scalar product in R^n) is also an n.c.i.

In what follows it is assumed everywhere that $f(t, 0) = 0$, $f(t, x) = f(t, -x)$ (although results similar to those formulated in Section 3 are also valid without this assumption). Then, obviously, $f(t, x) \geq 0$, and f^* has the same properties.

Let $x_t : T \rightarrow R^n$. Put

$$I_f(x_t) = \int_T f(t, x_t) d\mu,$$

$$\text{dom } I_f = \{x_t \in L_1 \mid I_f(x_t) < \infty\}.$$

Definition 1.

$$L_f = \bigcup_{n=1}^{\infty} n \text{ dom } I_f.$$

L_f is a linear space; this follows directly from the convexity and symmetry of $\text{dom } I_f$.

Definition 2. We say that $f(t, x)$ satisfies a) condition (A), if $f(t, x)$ is summable for every x from some neighborhood of zero in R^n ; b) condition (B), if $f(t, x)$ is summable for every $x \in R^n$.

Proposition 1. If f and f^* satisfy condition (A), then L_f and L_{f^*} are dual with respect to the bilinear form

$$\langle x_t, y_t \rangle = \int_T (x_t, y_t) d\mu.$$

In what follows, condition (A) is assumed to be satisfied. Put

$$B_f = \{x_t \in \text{dom } I_f \mid I_f(x_t) \leq 1\},$$

$$S_f = B_{f^*}^0 = \{x_t \in L_f \mid \langle x_t, y_t \rangle \leq 1 \forall y_t \in B_{f^*}\}.$$

Proposition 2. The sets B_f and S_f are absorbing in L_f , closed and bounded in the topology $\sigma(L_f, L_{f^*})$, and moreover

$$S_f \subset B_f \subset 2S_f$$

(with regard to the last relation, see (7)).

Therefore B_f and S_f generate equivalent norms in L_f , which, by analogy with Orlicz spaces, are naturally called the Luxemburg and Orlicz norms, respectively. It is easy to prove that L_f is complete with respect to the introduced norms.

Denote by E_f the closure, in the normed topology of L_f , of the set of bounded functions.

Definition 3. A linear continuous functional λ^* on L_f is called: a) singular, if for every $\varepsilon > 0$ there exists a set G_ε , $\mu G_\varepsilon < \varepsilon$, such that $\langle x_t, \lambda \rangle = 0$ for every

$x_t \in L_f$ equal to zero on G_ε (8); b) completely singular, if $\langle x_t, \lambda^* \rangle = 0$ for every $x_t \in E_f$; c) boundedly singular, if it is singular and

$$\|\lambda^*\| = \sup_{x_t \in L_f} \frac{\langle x_t, \lambda^* \rangle}{\|x_t\|} = \sup_{x_t \in E_f} \frac{\langle x_t, \lambda^* \rangle}{\|x_t\|}.$$

Proposition 3. A completely singular functional is singular.

Denote the set of completely singular functionals by $\tilde{\Lambda}^*$, and the boundedly singular ones by Λ^* . We shall also use the latter symbol to denote the set of restrictions of boundedly singular functionals to E_f .

Theorem 1.

$$E_f^* = L_{f^*} \oplus \Lambda^*,$$

$$L_f^* = L_{f^*} \oplus (\Lambda^* + \tilde{\Lambda}^*).$$

If, however, f satisfies condition (B), and only in this case,

$$E_f^* = L_{f^*}, \quad L_f^* = L_{f^*} \oplus \tilde{\Lambda}^*.$$

Let us note that I_{f^*} can be extended to all of L_f^* in the following way:

$$I_{f^*}(l^*) = \sup_{x_t \in L_f} (\langle x_t, l^* \rangle - I_f(x_t)).$$

Moreover, if $l = y_t \oplus \lambda^*$, where λ^* is a singular functional, then

$$I_{f^*}(l^*) = I_{f^*}(y_t) + I_{f^*}(\lambda^*), \quad I_{f^*}(\lambda^*) = \sup \langle x_t, \lambda^* \rangle, \quad x_t \in \text{dom } I_f,$$

and if L_f is considered together with the Orlicz norm, then the unit sphere of L_f^* is the set

$$\{l^* \in L_f^* \mid I_{f^*}(l^*) \leq 1\}.$$

2. In this section we formulate the problem for the sake of which the present investigation was undertaken. Let T be a bounded domain in R^m with boundary ∂T , μ the Lebesgue measure. If $\alpha = (\alpha_1, \dots, \alpha_m)$ is an integer-valued m -index (i.e., $\alpha_i \geq 0$ and integral), then the symbol D_α will denote the operator

$$D_\alpha = \partial^{|\alpha|} / \partial t_1^{\alpha_1} \dots \partial t_m^{\alpha_m},$$

where $|\alpha| = \sum \alpha_i$.

Suppose further that some set of n m -indices $A = (\alpha^1, \dots, \alpha^n)$ is given. Put $D_A = (D_{\alpha^1}, \dots, D_{\alpha^n})$, $|A| = \max_{1 \leq J \leq n} |\alpha^J|$, and for every real function u_t of class $W_\infty^{|A|}$,

$$D_A u_t = (D_{\alpha^1} u_t, \dots, D_{\alpha^n} u_t).$$

If α and β are two m -indices, then we write $\alpha < \beta$ under the condition that $\alpha_i \leq \beta_i$ and $|\alpha| < |\beta|$, and $\alpha < A$ when there exists $J (1 \leq J \leq n)$ such that $\alpha < \alpha^J$.

Finally, suppose that for each index $\alpha < A$ a real function r_t^α is given on ∂T . Our problem consists in the following:

$$\text{minimize } I_f(D_A u_t) \tag{1}$$

over the totality of all real functions u_t of class $W_\infty^{|A|}$, satisfying on ∂T the prescribed boundary conditions

$$D_\alpha u_t|_{\partial T} = r_t^\alpha \quad (\alpha < A). \tag{2}$$

Such functions will henceforth be called admissible. It is obvious

Proposition 4. *Let x_t be a bounded function on T with values in R^n . For there to exist an admissible function u_t such that $x_t = D_A u_t$, it is necessary and sufficient that for every function $y_t : T \rightarrow R^n$ of class $C^{|A|}$, $y_t = (y_t^1, \dots, y_t^n)$, satisfying the equation*

$$\sum_{J=1}^n (-1)^{|\alpha^J|} D_{\alpha^J} y_t^J = 0, \tag{3}$$

and for an arbitrary admissible \bar{u}_t , the equality

$$\langle x_t - D_A \bar{u}_t, y_t \rangle = 0 \tag{4}$$

hold.

If by M we denote the linear manifold of solutions of (3), then equality (4) determines a shifted linear manifold parallel to the orthogonal complement of M .

- Thus, problem (1) with condition (2) may be regarded as a particular case of the following general problem. Let M be a linear manifold in E_{f^*} , M^0 its annihilator in $E_{f^*} = L_f \oplus \Lambda$, $S = E_{f^*} / M^0$ the quotient space of E_{f^*} by M^0 , and $\gamma : E_{f^*} \rightarrow S$ the natural projection. It is required

$$\text{to minimize } I_f(l) \tag{5}$$

under the conditions

$$l \in E_{f^*}, \quad \gamma(l) = \xi, \tag{6}$$

where ξ is a fixed element of S .

Proposition 5. *If $\gamma^{-1}(\xi) \cap \text{int dom } I_f \neq \emptyset$, then*

$$\inf_{l \in \gamma^{-1}(\xi)} I_f(l) = \inf_{x_t \in \gamma^{-1}(\xi) \cap L_f} I_f(x_t).$$

(The interior of $\text{dom } I_f$ is considered with respect to the strong topology of L_f .)

As applied to problem (1), (2), the meaning of this proposition is the following. If M is defined by condition (3), then from the existence of at least one admissible u_t such that $D_A u_t$ belongs to the interior of $\text{dom } I_f$, it follows that

that the lower bound will not change if one passes from the set of admissible functions to all of $\gamma^{-1}(\xi)$.

Denote by M^{00} the closure of M in $\sigma(L_{f^*}, E_{f^*})$. For every $\xi \in S$, $y_t \in M^{00}$, set

$$\langle \xi, y_t \rangle = \langle l, y_t \rangle,$$

where $l \in \gamma^{-1}(\xi)$ is arbitrary. (This definition, obviously, does not depend on the choice of l .) We shall call the problem

$$\text{minimize } \langle \xi, y_t \rangle - I_{f^*}(y_t) \tag{7}$$

$$\text{subject to } y_t \in M^{00} \tag{8}$$

dual to problem (5), (6).

Theorem 2. 1) The lower bound in problem (5), (6) is attained.

- 2) If f satisfies condition (B) and $\gamma^{-1}(\xi) \cap \text{int dom } I_f \neq \emptyset$, then solutions exist in both problems. Moreover,
 - a) the minimum in (5), (6) is equal to the maximum in (7), (8);
 - b) if y_t^0 is a solution of problem (7), (8), then in order that $l^0 = x_t^0 \oplus \lambda^0$ ($\lambda^0 \in \Lambda$) be a solution of problem (5), (6), it is necessary and sufficient that the following three conditions be fulfilled:

$$f(t, x_t^0) + f^*(t, y_t^0) = \langle x_t^0, y_t^0 \rangle \quad \text{almost everywhere on } T,$$

$$\langle x_t^0 + \lambda^0, y_t \rangle = \langle \xi, y_t \rangle \quad \text{for all } y_t \in M,$$

$$\langle \xi, y_t^0 \rangle - \langle x_t^0, y_t^0 \rangle \geq \langle \lambda^0, y_t \rangle \quad \text{for all } y_t \in E_{f^*} \cap \text{dom } I_{f^*};$$

- c) if there exists a sequence $\{y_t^k\} \subset E_{f^*}$ converging to y_t^0 in $\sigma(L_{f^*}, E_{f^*})$ and a number $\varepsilon > 0$ such that, for every $y \in R^n$ with $|y| < \varepsilon$ ($|\cdot|$ is the Euclidean norm), $y_t^k + y \in \text{dom } I_{f^*}$ for all $k = 1, 2, \dots$, then $\lambda^0 = 0$ and, consequently, the lower bound in problem (5), (6) is attained in L_f .

Returning to problem (1), (2), one should note that here, too, the singular elements in the solution can apparently be interpreted as discontinuities, as was done in (2) for one-dimensional problems.

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