ON THE CONTINUITY OF THE METRIC PROJECTION ON CERTAIN CLASSES OF SUBSPACES IN A BANACH SPACE

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 513.88:513.83

MATHEMATICS

E. B. OSMAN

ON THE CONTINUITY OF THE METRIC PROJECTION ON CERTAIN CLASSES OF SUBSPACES IN A BANACH SPACE

(Presented by Academician A. N. Tikhonov on 27 IV 1970)

Let E and F be metric spaces, and let $K(F)$ be the set of all closed subsets of the space F. A mapping $\varphi : E \to K(F)$ is called a multivalued mapping from E into F. φ is called lower semicontinuous ([3]) if the set $\{x \in E : \varphi_x \subset G\}$ is open in E for every open subset $G \subset F$, and H-upper semicontinuous ([4,5]) if for every $x \in E$ the set φ_x is nonempty and the condition $x_n \to x$ implies the condition

$$\tilde{\rho}(\varphi_{x_n}, \varphi_x) = \sup\{\rho(y, \varphi_x) : y \in \varphi_{x_n}\} \to 0.$$

Both definitions coincide with the usual definition of continuity if φ is single-valued. If the mapping φ is upper semicontinuous and $\varphi_x \neq \emptyset$ for every $x \in E$, then φ will be H-upper semicontinuous; moreover, if φ_x is compact for every $x \in E$, then the converse is also true ([5]).

Let M be a set in E. The multivalued mapping T_M, which assigns to each point $x \in E$ the set

$$T_M x = \{y \in M : \rho(x, y) = \rho(x, M)\},$$

is called the metric projection of E onto M ([2]). The set M is called a set of existence ([1]) if $T_M x \neq \emptyset$ for every $x \in E$, and Chebyshev ([1]) if $T_M x$ is a singleton for every $x \in E$.

Everywhere in what follows: X is a B-space over the field of real numbers; $S = \{x \in X : \|x\| = 1\}$, $S^* = \{f \in X^* : \|f\| = 1\}$; $x_n \rightharpoonup x_0$ if the sequence $\{x_n\}$ converges weakly to x_0; $\overline{\mathrm{sp}}\{f_n\}$ is the closed linear span of the sequence $\{f_n\} \subset X^*$; $\dim \mathrm{sp}\{f_n\}$ is the dimension of the subspace $\mathrm{sp}\{f_n\}$.

Theorem 1. *In a reflexive B-space X the following assertions are equivalent:*

a) *the metric projection onto every closed subspace is lower semicontinuous;*

b) *X satisfies the following condition (L): from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^{\infty} \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $\|x_n - x_0\|_{\mathrm{sp}\{f_n\}} \to 0$, where*

$$\|x_n - x_0\|_{\mathrm{sp}\{f_n\}} = \sup\{f(x_n - x_0) : f \in \mathrm{sp}\{f_n\} \cap S^*\},$$

it follows that $x_n \to x_0$*;*

c) X *satisfies the following condition (L.1): from* $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^{\infty} \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $\rho(x_n - x_0, L) \to 0$, *where* $L = \bigcap_{n=0}^{\infty}\{x : f_n(x) = 0\}$, *it follows that* $x_n \to x_0$.

Corollary 1. *In a reflexive* B*-space* X *satisfying condition (L) (or, what is the same, condition (L.1)), the metric projection onto every Chebyshev subspace is continuous.*

Theorem 2. *In order that in a reflexive* B*-space* X *the metric projection onto every closed subspace of finite defec-*

then (respectively, onto every closed subspace of defect $\leq k$) is upper semicontinuous, it is necessary and sufficient that X satisfy the following condition (L^∞): from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^{\infty} \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $\dim \mathrm{sp}\{f_n\} < \infty$ it follows that $x_n \to x_0$ (respectively, that X satisfy the following condition L^k: from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^{\infty} \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $\dim \mathrm{sp}\{f_n\} \leq k$ it follows that $x_n \to x_0$).

Corollary 2. In a reflexive B-space X satisfying condition (L^∞) (respectively, satisfying condition (L^k)), the metric projection onto every Chebyshev subspace of finite defect (respectively, onto every Chebyshev subspace of defect $\leq k$) is continuous.

Theorem 3. In order that, in a B-space X, every closed subspace be Chebyshev with a continuous metric projection, it is necessary and sufficient that X be reflexive and satisfy the following condition (L_R): from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^{\infty} \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $\|x_n - x_0\|_{\mathrm{sp}\{f_n\}} \to 0$ it follows that $x_n \to x_0$ (or, what is the same, condition

$$(L_R.1): \quad \text{from } \{x_n\} \subset S,\ x_0 \in S,\ \{f_n\}_{n=0}^{\infty} \subset S^*,\ f_n(x_n) = f_0(x_0) = 1,\ \rho(x_n - x_0, L) \to 0,$$

where

$$L = \bigcap_{n=0}^{\infty}\{x :\ f_n(x) = 0\},$$

it follows that $x_n \to x_0$).

Theorem 4. In order that, in a strictly convex and reflexive B-space X, the metric projection onto every closed subspace of finite defect (respectively, onto every closed subspace of defect $\leq k$) be continuous, it is necessary and sufficient that X satisfy condition (L^∞) (respectively, that X satisfy condition (L^k)).

Theorem 5. In a reflexive B-space the metric projection onto every closed hyperplane is H-upper semicontinuous (in a strictly convex space, continuous).

Theorem 6. In a B-space X satisfying condition (L), the metric projection onto every reflexive subspace is H-upper semicontinuous. In particular, the metric projection onto every reflexive Chebyshev subspace is continuous.

A B-space X is called compactly rotund ([4,5]) (respectively, weakly compactly rotund ([4,5])) if for every $f \in S^*$ the set $\{x : f(x) = 1\} \cap S$ is either empty or compact (respectively, it is either empty or weakly compact).

Theorem 7. In a weakly compactly rotund B-space X satisfying condition (L), the metric projection onto every reflexive subspace is upper semicontinuous.

Theorem 8. In a B-space X, the metric projection onto every hyperplane that is an existence set is H-upper semicontinuous.

Theorem 9. In order that, in a B-space X, the metric projection onto every closed hyperplane be upper semicontinuous, it is necessary and sufficient that X be compactly rotund.

Example. Consider, in the space l_2 of real numerical sequences $x = \{\xi_i\}$ summable with square, the following norm, equivalent to the original one:

$$\|x\|_{S'} = \inf\{|\lambda| : x \in \lambda S'\},$$

where

$$S' = \left\{ x = \{\xi_i\} \in l_2 : \sum_{i=1}^{\infty} \frac{\xi_i^2}{i^2} \le 1, \quad \sum_{i=1}^{\infty} \xi_i^2 \le 3 \right\}.$$

Denote by X the space l_2 with the norm $\|x\|_{S'}$. It is not difficult to see that X is reflexive, strictly convex, and satisfies condition (L^∞). Con-

Thus, by virtue of Theorem 4, in the space X the metric projection onto each closed subspace of finite defect is discontinuous. It can be shown, however, that in X the metric projection onto some convex closed set is discontinuous.

Ural State University
named after A. M. Gorky
Sverdlovsk

Received
22 IV 1970

REFERENCES

[1] N. V. Efimov, S. B. Stechkin, DAN, 118, No. 1, 17 (1958).
[2] V. L. Klee, Math. Ann., 142, 292 (1961).

[3] E. Michael, Trans. Am. Math. Soc., 71, 152 (1951).
[4] E. V. Oshman, DAN, 185, No. 1, 34 (1969).
[5] E. V. Oshman, Mat. sborn., 80, No. 2, 181 (1969).

Note: Figure translations are in progress. See original paper for figures.