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Abstract

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MATHEMATICS

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EQUIMORPHISMS AND QUASICONFORMAL MAPPINGS OF THE ABSOLUTE

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It is known that every equimorphism $f : H^n \rightarrow 'H^n$ of Lobachevsky spaces induces a topological mapping $\varphi : \Sigma^{n-1} \rightarrow '\Sigma^{n-1}$ of their infinitely distant spheres ⁽¹⁾, and that, in this case, φ cannot be an arbitrary topological mapping ⁽²⁾. V. A. Efremovich posed the problem: to find a necessary and sufficient condition for the existence of an equimorphism $f : H^n \rightarrow 'H^n$ inducing a given mapping $\varphi : \Sigma^{n-1} \rightarrow '\Sigma^{n-1}$. In this paper it is proved that such a condition is the quasiconformality of the mapping φ .

What will be said below about notation remains valid if the symbols o, H^n, Σ^{n-1} are replaced by the symbols $'o, 'H^n$, and $'\Sigma^{n-1}$.

We shall denote by o a certain fixed point of the space H^n , and if $\xi_1, \xi_2 \in \Sigma^{n-1}$, then by the distance $\xi_1\xi_2$ we shall mean the angle $\angle \xi_1 o \xi_2$. Points of the space H^n will be denoted by small Latin letters, and points of the sphere Σ^{n-1} by small Greek letters. If $o \neq z \in H^n$, then the ray issuing from o and passing through the point z will be denoted by Γ_z , and the infinitely distant point to which the ray Γ_z leads will be denoted by the letter ζ , i.e. by the Greek letter corresponding to the Latin letter entering into the notation of the point through which such a ray passes. If $\xi \in \Sigma^{n-1}$, then the ray issuing from o and leading to the point ξ will be denoted by Γ_ξ . If $x, y \in H^n$, then by xy we shall denote the distance from x to y . If $\zeta \in \Sigma^{n-1}$ and $\rho > 0$, then by $\Omega(\zeta, \rho)$ we shall denote the $(n-2)$ -dimensional sphere of the space Σ^{n-1} of radius ρ with center at the point ζ .

Theorem. *If the infinitely distant sphere Σ^{n-1} of the n -dimensional Lobachevsky space H^n is mapped by means of a topological mapping φ onto the infinitely distant sphere $'\Sigma^{n-1}$ of another n -dimensional Lobachevsky space $'H^n$, then an equimorphism $f : H^n \rightarrow 'H^n$ inducing the mapping $\varphi : \Sigma^{n-1} \rightarrow '\Sigma^{n-1}$ exists if and only if φ is a quasiconformal mapping, $n = 2, 3, \dots$*

Proof. I. Necessity. Suppose that an equimorphism f with the properties indicated in the theorem exists. Let $f(o) = 'o$. Denote by ξ_1 and ξ_2 two arbitrarily chosen points of the sphere $\Omega = \Omega(\zeta, \rho)$ small enough for the constructions described below to be possible.

By virtue of the properties of equimorphisms of hyperbolic spaces ⁽¹⁾, there exists a number k greater than one such that, for any ray $\Gamma \subset H^n$, the distance between any point $x \in f(\Gamma)$ and a ray $A \subset 'H^n$ having the same origin and the same infinitely distant point as the image $f(\Gamma)$ of the ray Γ , is less than k , and the distance between $f(\Gamma)$ and any point of the ray A is less than k .

By virtue of the properties of equimorphisms of geodesic spaces ⁽³⁾, there is a number $\lambda = \lambda(1)$, greater than one, such that if $a_1, a_2 \in H^n$ and $a_1 a_2 > 1$, then

$$\frac{1}{\lambda} a_1 a_2 < f(a_1) f(a_2) < \lambda a_1 a_2.$$

Denote by z that point of the ray Γ_ζ for which $z x_1 = 4\lambda k$, where x_1 is the foot of the perpendicular dropped from the point z to the ray Γ_{ξ_1} . Denote $\varphi(\zeta), \varphi(\xi_1)$, and $\varphi(\xi_2)$ by ζ', ξ'_1 , and ξ'_2 . Denote by z^* the foot of the perpendicular dropped from the point $f(z)$ to the ray $\Gamma_{\zeta'}$, and denote by x_1^* the foot of the perpendicular dropped from the point $f(x_1)$ to the ray $\Gamma_{\xi'_1}$. Denote by x_2^* the foot of the perpendicular dropped from z^* to the ray $\Gamma_{\xi'_2}$, and denote by x_2 that point of the ray Γ_{ξ_2} such that the distance $\rho(x_2^*, f(\Gamma_{\xi_2})) = x_2^* f(x_2)$. It is easy to see that

$$\rho(z^*, \Gamma_{\xi'_1}) \leq z^* x_1^* \leq z^* f(z) + f(z) f(x_1) + f(x_1) x_1^* < 4\lambda^2 k + 2k;$$

$$\rho(z^*, \Gamma_{\xi'_2}) = z^* x_2^* \geq f(z) f(x_2) - f(z) z^* - f(x_2) x_2^* > 4k - k - k = 2k.$$

Considering the right triangle $oz^* x_2^*$ and the right triangle whose vertices are o, z^* , and the foot of the perpendicular dropped from the point z^* to the ray $\Gamma_{\xi'_1}$, we find that

$$\sin \zeta' \xi'_1 = \text{sh } \rho(z^*, \Gamma_{\xi'_1}) / \text{sh } oz^* \quad \text{and} \quad \sin \zeta' \xi'_2 = \text{sh } \rho(z^*, \Gamma_{\xi'_2}) / \text{sh } oz^*. \quad (1)$$

Therefore

$$\sin \zeta' \xi'_1 / \sin \zeta' \xi'_2 = \text{sh } \rho(z^*, \Gamma_{\xi'_1}) / \text{sh } \rho(z^*, \Gamma_{\xi'_2}).$$

If the sphere Ω is sufficiently small, then therefore $\zeta' \xi' \leq 2 \sin \zeta' \xi'_1$. Taking into account the last inequality and the inequalities $\rho(z^*, \Gamma_{\xi'_1}) < 4\lambda^2 k + 2k$, $\rho(z^*, \Gamma_{\xi'_2}) > 2k$, we obtain

$$\zeta' \xi'_1 / \zeta' \xi'_2 < \zeta' \xi'_1 / \sin \zeta' \xi'_2 < 2 \sin \zeta' \xi'_1 / \sin \zeta' \xi'_2 = 2 \text{sh } \rho(z^*, \Gamma_{\xi'_1}) / \text{sh } \rho(z^*, \Gamma_{\xi'_2}) < 2 \text{sh}(4\lambda^2 k + 2k) / \text{sh } 2k.$$

Denoting $2 \text{sh}(4\lambda^2 k + 2k) / \text{sh } 2k$ by c , we obtain:

$$\zeta' \xi'_1 / \zeta' \xi'_2 < c. \quad (2)$$

It follows from (2) that φ is a quasiconformal mapping.

Thus, if the mapping $\varphi : \Sigma^{n-1} \rightarrow ' \Sigma^{n-1}$ is induced by the equimorphism $f : H^n \rightarrow ' H^n$, then φ is a quasiconformal mapping. Independently of the author of the present work and simultaneously with him, this result was obtained by G. A. Margulis.

II. Sufficiency. To the notation introduced earlier we add the following notation. If $z \in H^n$, then we shall denote by P_z the number e^{-oz} and by Ω_z the sphere $\Omega(\zeta, P_z)$. Instead of the notations $f(z)$, $f(Z)$, $\varphi(\zeta)$, $\varphi(Z)$, we shall use the notations z' , Z' , ζ' , and Z' , if $z \in H^n$, $\zeta \in \Sigma^{n-1}$, $z \subset H^n$, and $Z \subset \Sigma^{n-1}$. The maximum of distances of the form $\zeta'\xi'$, where $\xi \in \Omega_z$, will be denoted by P'_z , if $z \in H^n$. Suppose that $\varphi : \Sigma^{n-1} \rightarrow \Sigma^{n-1}$ is a quasiconformal mapping. By virtue of the properties of quasiconformal mappings,* there exist numbers M, K , and Q , greater than one, such that if $\zeta \in \Sigma^{n-1}$, $\xi_1 \in \Sigma^{n-1}$, $\xi_2 \in \Sigma^{n-1}$, and $\zeta\xi_1 < \zeta\xi_2 < 1/Q$, then

$$\frac{1}{M} \left(\frac{\zeta\xi_1}{\zeta\xi_2} \right)^K < \frac{\zeta'\xi'_1}{\zeta'\xi'_2} < M \left(\frac{\zeta\xi_1}{\zeta\xi_2} \right)^{1/K}. \quad (3)$$

Define a mapping f of the space $H^n \setminus \mathfrak{A}$ into the space $'H^n$ by means of the following two conditions \mathcal{A} and \mathcal{B} . If $z \in H^n \setminus \mathfrak{A}$, then: \mathcal{A}) $e^{-'oz'} = P'_z$. \mathcal{B}) $\Gamma_{z'} = \Gamma_{\zeta'}$.

Here $\mathfrak{A} \subset H^n$ is an n -dimensional closed ball with center at the point o , whose radius R exceeds one and is so large that if $z \in H^n \setminus \mathfrak{A}$, then $P_z < 1/Q$, $P_z < \pi/6$, and $P'_z < \pi/6$.

It is easy to see that the mapping f^{-1} of the space $(H^n \setminus \mathfrak{A})'$ onto the space $H^n \setminus \mathfrak{A}$ is a one-to-one mapping. We shall prove that f is a uniformly continuous mapping of the space $H^n \setminus \mathfrak{A}$ onto the space $(H^n \setminus \mathfrak{A})'$.

* This property, described by inequality (3), was essentially proved, however for the two-dimensional case, by Gehring (4). Its extension to the multidimensional case causes no difficulties.

Let $a, b \in H^n \setminus \mathfrak{A}$, $ab < 1/12$ and $oa \geq ob$. Denote $\rho(a, \Gamma_b)$ by d_1 , $oa - ob$ by d_2 , $\angle aob$ by Δ_1 , and $P_b - P_a$ by Δ_2 .

Since $d_1 \leq ab < 1/12$ and $oa > R > 1$, it follows that $\Delta_1 < \pi/6$, for both on H^2 and on E^2 the acute angle of a right triangle lying opposite a leg smaller than half the hypotenuse is less than $\pi/6$. Considering the right triangle whose vertices are o, a , and the foot of the perpendicular dropped from a to the ray Γ_b , we find that

$$\text{sh } d_1 = \text{sh } oa \cdot \sin \Delta_1 = \frac{\sin \Delta_1}{2P_a} (1 - P_a^2).$$

Since $\Delta_1 < \pi/6$, $P_a < \pi/6$, and $d_1 < 1/12$, we have $\text{sh } d_1 < 2d_1$, $\sin \Delta_1 > 3/4\Delta_1$, and $(1 - P_a^2) > 2/3$,

$$2d_1 < \frac{\Delta_1}{P_a} < 10d_1 \leq 10ab. \quad (4)$$

From the inequality $d_2 \leq ab < 1/12$ it follows that $d_2 < e^{d_2} - 1 < 2d_2$ and

$e^{d_2} < 2$. Since

$$d_2 = oa - ob = \ln(1/P_a) - \ln(1/P_b) = \ln P_b/P_a; \quad e^{d_2} = P_b/P_a;$$

$$e^{d_2} - 1 = (P_b - P_a)/P_a = \Delta_2/P_a,$$

the inequality

$$d_2 < \Delta_2/P_a < 2d_2 \leq 2ab \quad (5)$$

is valid.

We now define the number Δ_3 in the following way. If $P_{b'} \geq P_{a'}$, then denote by ξ the point of the sphere Ω_a nearest to the point $\eta \in \Omega_b$ such that $\eta'\beta' = P_{b'}$, and if $P_{a'} > P_{b'}$, then denote by ξ the point of the sphere Ω_b nearest to the point $\eta \in \Omega_a$ such that $\eta'\alpha' = P_{a'}$. We denote the distance $\eta\xi$ by Δ_3 . It is not hard to show that $\Delta_3 \leq \Delta_1 + \Delta_2$. Taking into account (4), (5), and the inequalities $P_b \geq P_a$, $ab < 1/12$, we obtain from this that $\xi\eta/P_a < 12ab < 1$ and $\xi\eta/P_b < 12ab < 1$. It also follows from (4) that $\alpha\beta < P_a$. Then also $\alpha\beta < P_b$. Since, moreover, $P_a < 1/Q$ and $P_b < 1/Q$, it follows from inequality (3) that

$$\alpha'\beta'/P_{a'} < M(\alpha\beta/P_a)^{1/K}, \quad \alpha'\beta'/P_{b'} < M(\alpha\beta/P_b)^{1/K},$$

$$\xi'\eta'/P_{a'} < M(\xi\eta/P_a)^{1/K}, \quad \xi'\eta'/P_{b'} < M(\xi\eta/P_b)^{1/K}.$$

In view of the inequalities

$$\alpha\beta/P_b < \alpha\beta/P_a < 10ab, \quad \xi\eta/P_b < \xi\eta/P_a = \Delta_3/P_a < \Delta_1/P_a + \Delta_2/P_a < 12ab,$$

it follows that

$$\alpha'\beta'/P_{a'} < M(10ab)^{1/K}, \quad \alpha'\beta'/P_{b'} < M(10ab)^{1/K}, \quad \xi'\eta'/P_{a'} < M(12ab)^{1/K}, \quad \xi'\eta'/P_{b'} < M(12ab)^{1/K}.$$

In the case where $P_{b'} \geq P_{a'}$, from the last inequalities, taking into account that

$$P_{b'} = \beta'\eta' \leq \beta'\alpha' + \alpha'\xi' + \xi'\eta' \leq \beta'\alpha' + P_{a'} + \xi'\eta',$$

we obtain the inequality

$$P_{b'}/P_{a'} < 1 + M(10ab)^{1/K} + M(12ab)^{1/K} < 1 + 2M(12ab)^{1/K}.$$

In the case where $P_{a'} > P_{b'}$, taking into account that

$$P_{a'} = \alpha'\eta' \leq \alpha'\beta' + \beta'\xi' + \xi'\eta',$$

we analogously obtain the inequality

$$P_{a'}/P_{b'} < 1 + 2M(12ab)^{1/K}.$$

Denote $|'oa' - 'ob'|$ by d'_2 . It is easy to compute that if $'oa' \geq 'ob'$, then $d'_2 = \ln(P_{b'}/P_{a'})$, and if $'oa' < 'ob'$, then $d'_2 = \ln P_{a'}/P_{b'}$. In both cases

$$d'_2 < \ln(1 + 2M(12ab)^{1/K}).$$

Denote $\rho(a', \Gamma_{b'})$ by d'_1 . The ball \mathfrak{A} was chosen so large that from the condition $z \in H^n \setminus \mathfrak{A}$ it follows that $P_z < \pi/6$. Since $a \in H^n \setminus \mathfrak{A}$ and $\alpha\beta < 10abP_a < P_a$, as is easy to see, it follows that $P_{a'} < \pi/6$ and $\alpha'\beta' < \pi/6$. Considering the right triangle whose vertices are $'o, a'$ and the foot of the perpendicular dropped from a' to the ray $\Gamma_{b'}$, we find

$$\operatorname{sh} d'_1 = \operatorname{sh}' oa' \cdot \sin \alpha' \beta' = \frac{\sin \alpha' \beta'}{2P_{a'}} (1 - (P_{a'})^2). \quad (6)$$

It follows from this that

$$d'_1 < \frac{1}{2} \alpha' \beta' / P_{a'} < \frac{1}{2} M(10ab)^{1/K}.$$

It is easy to see that

$$a'b' \leq 2d'_1 + d'_2$$

and therefore

$$a'b' < M(10ab)^{1/K} + \ln(1 + 2M(12ab)^{1/K}).$$

The last inequality shows that f is a uniformly continuous mapping of the space $H^n \setminus \mathfrak{A}$ onto the space $(H^n \setminus \mathfrak{A})'$.

Passing to the proof of the uniform continuity of the mapping f^{-1} , we shall prove that the inequality

$$a'b' > \min \left(\operatorname{arsh} \left(\frac{1}{8M} \left(\frac{ab}{11} \right)^K \right), \operatorname{arsh} \left(\frac{1}{16M} \left(\frac{ab}{4} \right)^K \right), \ln \left(1 + \frac{1}{2M} \left(\frac{ab}{4} \right)^K \right) \right). \quad (7)$$

Suppose that $d_1 > d_2/20$. Then $2d_1 + d_2 > ab$ and $d_1 > ab/22$. From (4) in this case it follows that $\Delta_1/P_a = \alpha\beta/P_a > ab/11$. Taking (3) into account, we obtain

$$\frac{\alpha' \beta'}{P_{a'}} > \frac{1}{M} \left(\frac{ab}{11} \right)^K.$$

From the inequalities mentioned earlier $\alpha'\beta' < \pi/6$ and $P'_a < \pi/6$, it follows that $\sin \alpha' \beta' > \frac{1}{2} \alpha' \beta'$ and $1 - (P'_a)^2 > \frac{1}{2}$. Therefore, from (6) it follows that $\operatorname{sh} d'_1 > \alpha' \beta' / P'_a$. From the latter inequality, taking into account that $\operatorname{sh} \alpha' b' \geq \operatorname{sh} d'_1$ and

$$\frac{\alpha' \beta'}{P'_a} > \frac{1}{M} \left(\frac{ab}{11} \right)^K,$$

we obtain inequality (7).

Suppose now that $d_2 \geq 20d_1$. Taking (4) and (5) into account, we find that in this case

$$\Delta_2/P_a - \Delta_1/P_a > d_2 - 10d_1 \geq d_2 - 10d_2/20 \geq d_2/2,$$

$\Delta_2 - \Delta_1 = P_b - P_a - \alpha\beta > \frac{1}{2}d_2P_a > 0$; $\alpha\beta + P_a < P_b$. From the last inequality it follows that the sphere Ω_a lies inside* the sphere Ω_b and that therefore the image Ω'_a of the sphere Ω_a lies entirely inside the image Ω'_b of the sphere Ω_b . Let $\nu \in \Omega_a$ and $\alpha'\nu' = P'_a$. For every point $\xi \in \Omega_b$

$$\beta\alpha + \alpha\nu + \nu\xi = \beta\alpha + P_a + \nu\xi \geq \beta\xi = P_b;$$

$$\nu\xi \geq P_b - P_a - \alpha\beta > \frac{1}{2}d_2P_a.$$

Therefore the sphere $\Omega(\nu, d_2P_a/2)$ lies entirely inside the sphere Ω_b . Let $\tau \in \Omega(\nu, d_2P_a/2)$ and $\alpha'\nu' + \nu'\tau' = a'\tau'$. Let $\gamma \in \Omega_b$ and

$$\alpha'\gamma' = \alpha'\nu' + \nu'\tau' + \tau'\gamma'.$$

Since $ab \leq 2d_1 + d_2 \leq \frac{2}{20}d_2 + d_2 < 2d_2$, we have $d_2/2 > ab/4$, and consequently

$$\nu\tau/P_a = d_2P_a/2P_a > ab/4.$$

By virtue of (3),

$$\frac{\nu'\tau'}{\nu'\alpha'} = \frac{\nu'\tau'}{P'_a} > \frac{1}{M} \left(\frac{\nu\tau}{P_a} \right)^K > \frac{1}{M} \left(\frac{ab}{4} \right)^K.$$

Let us now consider two cases.

First case. $\alpha'\beta' \geq \frac{1}{2}\nu'\tau'$. Then

$$\frac{\alpha'\beta'}{P'_a} \geq \frac{\nu'\tau'}{2P'_a} > \frac{1}{2M} \left(\frac{ab}{4} \right)^K.$$

It was shown above that $\text{sh } d'_1 > \alpha'\beta'/8P'_a$, and evidently the last inequality remains valid independently of whether $d_1 > d_2/20$ or $d_2 \geq 20d_1$. Therefore in the case under consideration

$$\text{sh } d'_1 > \frac{1}{16M} \left(\frac{ab}{4} \right)^K$$

and inequality (7) is satisfied.

Second case. $\alpha'\beta' < \frac{1}{2}\nu'\tau'$. Considering the triangle α', β', γ' , we find that

$$P'_b \geq \beta'\gamma' > \alpha'\gamma' - \alpha'\beta' = P'_a + \nu'\tau' + \tau'\gamma' - \alpha'\beta' > P'_a + \nu'\tau' - \frac{1}{2}\nu'\tau' = P'_a + \nu'\tau'/2;$$

$$\frac{P'_b}{P'_a} > 1 + \frac{\nu'\tau'}{2P'_a} > 1 + \frac{1}{2M} \left(\frac{ab}{4} \right)^K.$$

Taking into account that in this case

$$a'b' \geq |oa' - ob'| = oa' - ob' = \ln P'_b/P'_a,$$

we obtain inequality (7).

It is now not difficult to show that the mapping $f : (H^n \setminus \mathfrak{A}) \rightarrow (H^n \setminus \mathfrak{A})'$ is an equimorphism. This equimorphism can be extended to an equimorphism $f : H^n \rightarrow 'H^n$, inducing the mapping $\varphi : \Sigma^{n-1} \rightarrow '\Sigma^{n-1}$.

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¹ V. A. Efremovich, E. S. Tikhomirova, *Izv. AN SSSR, ser. matem.*, 28, 5 (1964). ² V. A. Efremovich, V. I. Pupko, *DAN*, 160, No. 1 (1965). ³ V. A. Efremovich, *Uch. zap. Ivanovsk. ped. inst.*, 31 (1963). ⁴ F. W. Gehring, *Bull. Am. Math. Soc.*, 69, 2, 146 (1953).

* That is, it is contained in the smaller of the two domains into which the sphere Ω_{z_1} divides the sphere Σ^{n-1} .

Note: Figure translations are in progress. See original paper for figures.

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