SOLUTION OF SPATIAL PROBLEMS OF THE THEORY OF ELASTICITY BY MEANS OF ANALYTIC FUNCTIONS AND CONTOUR INTEGRALS FOR CERTAIN NON-AXISYMMETRIC BODIES

Theory of Elasticity

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Fig. 1

Figure 1: Fig. 1

Abstract

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SOLUTION OF SPATIAL PROBLEMS OF THE THEORY OF ELASTICITY BY MEANS OF ANALYTIC FUNCTIONS AND CONTOUR INTEGRALS FOR CERTAIN NON-AXISYMMETRIC BODIES

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By the superposition proposed in work ([1]), a connection is established between an axisymmetric state of an elastic space and plane states. In works ([2]–[5]), by means of two different superpositions, relations were established between the spatial states of bodies of revolution, both solid and containing cavities, on the one hand, and states of plane deformation and antiplane deformation of transverse sections of cylinders, on the other. With the aid of these relations and the known representations of the components of the plane state and antiplane deformation in terms of analytic functions of a complex variable, representations were obtained for the components of stresses and displacements in bodies of revolution through densities of integrals of the Cauchy type. With the aid of these representations, solutions were found for a number of problems of the theory of elasticity for such bodies ([6]–[8]).

In the present work, analogous representations are obtained for certain classes of bodies that do not possess axial symmetry. The superpositions carried out here for this purpose have the special feature that, in the process of their execution, all elements of the surface of the body are successively brought into coincidence with the lateral surface of a cylinder, i.e., with the boundary of the domain of analyticity of the functions introduced (where this requirement is not fulfilled, it may prove impossible to satisfy arbitrary boundary conditions).

Fig. 1

1. Let the body A be cut out from a cylinder B in such a way that the surfaces A and B are in contact over some area or curve (see Fig. 1a–

here the cylinder is cut away for clarity of the drawing). The coordinate axes $x_\|, y_\|, z_\|$ are associated with the cylinder B (the axis $y_\|$ is parallel to its generator), while x, y, z and r, θ, z are associated with the body A. We shall determine the position of the system $x_\|, y_\|, z_\|$ relative to x, y, z by the coordinates of the origin of the first system relative to the second (a_x, a_y, a_z) and by the cosines of the angles between the axes ($\lambda_{xy_\|} = \cos(x, y_\|)$, etc.).

Suppose that the body A can be cut out from this same cylinder B by changing, in a definite manner, the position of the latter relative to A.

(Fig. 1b). In the general case, this change is carried out by rotations and linear displacements of the cylinder relative to the axes x, y, z associated with A. In this case the displacements of the cylinder and the quantities a and λ associated with them are determined by certain functions, depending on the shape of the body, of the angle α between the axis x and OC—a line parallel to the projection of the axis $x_\|$ onto the plane xy (Figs. 1, 2). These functions are such that, for $0 \le \alpha \le 2\pi$, the surfaces A and B do not intersect, while the line (or area) of their contact, in the process of variation of α, moves over the surface A and, as α varies within the indicated limits, sweeps out the entire surface A.

In other words, we assume that the surface of the body A is the envelope of a family of cylindrical surfaces B under their displacements within the range of variation of α from 0 to 2π.

Let the cylinder B be in a state formed by superposition of plane strain ($\varepsilon_{y\|} = \tau_{y\|z\|} = \tau_{y\|x\|} = 0$) and antiplane deformation of the transverse sections ($\sigma_{x\|} = \sigma_{y\|} = \sigma_{z\|} = \tau_{x\|z\|} = 0$). We shall regard the displacement components of the cylinder as functions of a certain parameter, which may be taken equal to the angle α. In the cut-out body A, we pass from the displacement components in the coordinates $x_\|, y_\|, z_\|$ to the displacement components in the cylindrical coordinates r, θ, z, and superpose these components as α varies from 0 to 2π. As a result, we obtain a spatial state of the cut-out body, whose components are determined by certain integrals containing the displacements in plane strain and antiplane deformation.

Using the Kolosov—Muskhelishvili formulas and the solution of the torsion problem for a cylindrical rod, we represent the displacements in plane strain and antiplane deformation through three functions of the complex variable $\zeta = z_\| + ix_\|$ and of the parameter α. We shall assume that, with respect to α, these functions can be represented, for example, in the form

$$\varphi(\zeta, \alpha) = \sum_{n=-\infty}^{\infty} \varphi_n(\zeta) e^{-in\alpha}, \qquad \psi(\zeta, \alpha) = \sum_{n=-\infty}^{\infty} \psi_n(\zeta) e^{-in\alpha},$$

$$\Phi(\zeta, \alpha) = -i \sum_{n=-\infty}^{\infty} \Phi_n(\zeta) e^{-in\alpha}. \tag{1}$$

Here $\varphi_n(\zeta), \psi_n(\zeta), \Phi_n(\zeta)$ are functions analytic in the domain of the cross-section of the cylinder.

We shall represent the analytic functions $\varphi_n, \psi_n, \Phi_n$ by Cauchy-type integrals and obtain representations of the radial u, tangential v, and axial w displacement components for the spatial state of the body under consideration:

$$u = \frac{1}{8\mu\pi} \operatorname{Im} \sum_{n=-\infty}^{\infty} \int_L \{e^{i\theta} [\chi f_{1n}(t) I_1 + f'_{1n}(t) I_2 + f_{2n}(t) I_3 + f_{3n}(t) I_4] +$$

$$+ e^{-i\theta} [\chi f_{1n}(t) I_5 + f'_{1n}(t) I_6 + f_{2n}(t) I_7 + f_{3n}(t) I_8]\}\, dt,$$

$$v = \frac{1}{8\mu\pi} \operatorname{Re} \sum_{n=-\infty}^{\infty} \int_L \{e^{i\theta} [\chi f_{1n}(t) I_1 + f'_{1n}(t) I_2 + f_{2n}(t) I_3 + f_{3n}(t) I_4] -$$

$$- e^{-i\theta} [\chi f_{1n}(t) I_5 + f'_{1n}(t) I_6 + f_{2n}(t) I_7 + f_{3n}(t) I_8]\}\, dt,$$

$$w = \frac{1}{4\mu\pi} \operatorname{Im} \sum_{n=-\infty}^{\infty} \int_L [\chi f_{1n}(t) I_9 + f'_{1n}(t) I_{10} + f_{2n}(t) I_{11} + f_{3n}(t) I_{12}]\, dt.$$

Here

$$I_j = \int_0^{2\pi} P_j \frac{e^{-in\alpha}}{t - \zeta}\, d\alpha \qquad (j = 1, \dots, 12),$$

$$P_1 = -\overline{P}_7 = p_1 - ip_2, \quad P_2 = -\bar{\zeta}(p_1 - ip_2), \quad P_3 = -P_5 = -(p_1 - ip_2),$$

$$P_4 = -\overline{P}_8 = -(\lambda_{yy\|} + i\lambda_{xy\|}), \qquad P_6 = -\zeta(p_1 + ip_2), \qquad P_9 = -\overline{P}_{11} = \overline{P}_3,$$

$$P_{10} = -\bar{\zeta} p_3, \qquad P_{12} = -i\lambda_{zy\|}, \tag{2}$$

$$p_1 = \lambda_{xz\|} + i\lambda_{xx\|}, \qquad p_2 = \lambda_{yz\|} + i\lambda_{yx\|}, \qquad p_3 = \lambda_{zz\|} + i\lambda_{zx\|},$$

$$\zeta = (r\cos\theta - a_x)p_1 + (r\sin\theta - a_y)p_2 + (z - a_z)p_3,$$

f_{in} $(i = 1, 2, 3)$ are densities of Cauchy-type integrals, $\varkappa = 3 - 4\nu$, ν is Poisson' s ratio, μ is Lamé' s constant, $t = z_{0\|} + iz_{0\|}$ is the affix of a point of the contour L of the cross section of the cylinder B.

Fig. 2

Figure 2: Fig. 2

Fig. 2

Putting in (2)

$a_x = a_y = a_z = \lambda_{zx\|} = \lambda_{zy\|} = \lambda_{xz\|} = \lambda_{yz\|} = 0, \quad \lambda_{xx\|} = \lambda_{yy\|} = \cos\alpha, \quad \lambda_{xy\|} = -\lambda_{yx\|} = -\sin\alpha, \quad \lambda_{zz\|} = 1,$

we obtain representations for a body of revolution.

Analogously to (2), representations for stresses are obtained. We note that the representations (2) contain arbitrary constants, which make it possible, in solving the problem, to use techniques analogous to those proposed by D. I. Sherman for the plane problem and applied in (7) for the axisymmetric case.

2. Let us consider some special cases.

a) If in (2) we put

$a_x = a_y = \lambda_{xz\|} = \lambda_{yz\|} = \lambda_{zy\|} = \lambda_{zx\|} = 0, \quad \lambda_{xy\|} = -\lambda_{yx\|} = -\sin\alpha, \quad \lambda_{xx\|} = \lambda_{yy\|} = \cos\alpha, \quad \lambda_{zz\|} = 1, \quad a_z \neq$

in particular $a_z = k_0\alpha$, where $k_0 = \text{const}$, then the body A will be cut out of the cylinder B, the generators of which are parallel to the planes xy and, as the angle α changes, move along helical surfaces. In other words, the cylinder rotates about the z-axis with simultaneous displacement along it. In this way one can obtain a body having the form of a screw (under certain restrictions on the form of the thread in the meridional section). This case is shown in Fig. 2a. Here (as also below in Figs. 2b and 2c) the positions of the cylinder for two values of α are shown by solid and dash-dot lines; in the lower projection the body being cut out is shown by a dashed line. In this case, in expressions (2),

$$P_1 = P_3 = iP_4 = -\overline{P}_5 = -\overline{P}_7 = -i\overline{P}_8 = -ie^{i\alpha}, \qquad P_2 = -i\bar{\zeta}e^{i\alpha},$$
$$P_6 = -i\bar{\zeta}e^{-i\alpha},$$
$$P_9 = -P_{11} = 1, \qquad P_{10} = -\bar{\zeta}, \qquad P_{12} = 0, \qquad \zeta = z - a_z + ir\cos(\theta + \alpha).$$

b) If we put $a_z = \lambda_{xz\|} = \lambda_{yz\|} = \lambda_{zy\|} = \lambda_{zx\|} = 0$, $\lambda_{zz\|} = 1$, $\lambda_{xy\|} = -\lambda_{yx\|} = -\sin\alpha$, $\lambda_{xx\|} = \lambda_{yy\|} = \cos\alpha$, then we obtain the case shown in Fig. 2b. Here the cylinder B, whose generators are parallel to the plane xy, rotates about the z-axis with a simultaneous displacement in the plane xy. In this way, from the cylinder B one can cut out, for example, such a body A, whose sections in planes parallel to xy have the form of ellipses, ovals, etc.

In this case, in expressions (2) the values $P_1, \dots, P_{12}$ are obtained the same as in the preceding case, and

$$\xi = z + ir\cos(\theta + \alpha) - i(a_x\cos\alpha - a_y\sin\alpha).$$

Fig. 3

Figure 3: Fig. 3

c) For $a_y = \lambda_{zy\parallel} = 0$, $\lambda_{xx\parallel} = \cos\alpha\cos\gamma$, $\lambda_{yy\parallel} = \cos\alpha$, $\lambda_{xy\parallel} = \sin\alpha$, $\lambda_{yx\parallel} = -\sin\alpha\cos\gamma$, $\lambda_{zz\parallel} = \cos\gamma$, $\lambda_{zx\parallel} = \sin\gamma$, $\lambda_{xz\parallel} = -\sin\gamma\cos\alpha$, $\lambda_{yz\parallel} = \sin\alpha\sin\gamma$, we obtain the case shown in Fig. 2c. Here the generators of the cylinder are parallel to the plane xy. Rotation of the cylinder about the z-axis through the angle α is accompanied by its rotation about the $y_\parallel$-axis through the angle $\gamma(\alpha)$. In this way, from the cylinder one can cut out, for example, a worm screw (under certain restrictions on its shape in the meridional section). In this case, in expressions (2)

$$P_1 = -\overline{P}_7 = -ie^{i(\alpha-\gamma)}, \qquad P_2 = -i\overline{\xi}e^{i(\alpha+\gamma)},$$
$$P_3 = -\overline{P}_5 = -ie^{i(\alpha+\gamma)},$$
$$P_4 = -P_8 = -e^{i\alpha}, \qquad P_6 = -i\overline{\xi}e^{-i(\alpha-\gamma)}, \qquad P_9 = -P_{11} = e^{-i\gamma},$$
$$P_{10} = -\zeta e^{i\gamma}, \qquad P_{12} = 0,$$
$$\xi = [z - a_z + ir\cos(\alpha+\theta) - ia_x\cos\alpha]\, e^{i\gamma}.$$

Fig. 3

d) Various combinations of the displacements described in paragraphs 2a–c are possible. For example, superposing on the displacements of the cylinder described in paragraphs 2a–c a displacement in a plane parallel to xy, one can cut out a conical screw (Fig. 3), etc.

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Note: Figure translations are in progress. See original paper for figures.