SPACES GENERATED BY GENFUNCTIONS

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MATHEMATICS

I. V. Shragin

SPACES GENERATED BY GENFUNCTIONS

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1°. Let X be a space with a σ-finite complete measure μ ([1]), with $0 \leq \mu X \leq \infty$; let S be the set of all measurable functions on X with values in $\overline{R} = [-\infty, \infty]$; let S_f be the set of almost everywhere (a.e.) finite functions in S. As usual, functions that coincide a.e. are regarded as equal.

Definition. A function $M(u, x)$, $0 \leq u \leq \infty$, $x \in X$, with values in $[0, \infty]$, is called a **pregenfunction** if it is measurable in x for each u, nondecreasing and left-continuous (in the topology of $\overline{R}$) in u for almost every x, and the function $M(0, x)$ is summable on X.

Let M be a pregenfunction. Then, if $\varphi \in S$, we also have $M(|\varphi(\cdot)|, \cdot) \in S$.

Set

$$I_M\varphi = \int_X M[|\varphi(x)|, x]\, d\mu, \qquad P_M = \{\varphi \in S_f : I_M\varphi < \infty\}, \qquad P_M^\alpha =$$

$$= \{\varphi \in S_f : \alpha\varphi \in P_M\}, \qquad L^M = \bigcup_{\alpha>0} P_M^\alpha, \qquad L_M^f = \bigcap_{\alpha>0} P_M^\alpha.$$

Since the set P_M is convex and symmetric with respect to the zero point θ ($\theta(x) = 0$ a.e.), L^M is a vector space, and L_M^f is its subspace. It is obvious that $L_M^f = L^M$ if and only if $L^M = P_M$, i.e., when P_M itself is a vector space.

Set $d_M(x) = \sup\{u : M(u, x) < \infty\}$. It is known ([5]) that $d_M \in S$. One can show that $L^M = \{\theta\}$ if and only if $d_M = 0$; $L_M^f = \{\theta\}$ if and only if $d_M \in S_f$.

2°. In their works ([10,11]), Musielak and Orlicz introduced on the space L^M (which they considered as an example of an abstract modular space) the F-norm

$$\|\varphi\| = \inf\{\varepsilon > 0 : I_M(\varepsilon^{-1}\varphi) \leq \varepsilon\}, \tag{1}$$

satisfying the condition: $\lim \|\varphi_n\| = 0$ if and only if $\lim I_M(\alpha\varphi_n) = 0$ for every $\alpha > 0$ (any F-norm in L^M possessing this property will be called **normal**). In ([10,11]) it is assumed that the pregenfunction M takes finite values for all $(u, x) \in [0, \infty) \times X$, is continuous in u for each x, and $M(u, x) = 0$ if and only if $u = 0$. It turns out that these conditions can be weakened. To this end we give the following

Definition. A pregenfunction M is called a **genfunction** if $M(0, x) = 0$ and $M(\infty, x) > 0$ a.e. on X, while $M(+0, x) = 0$ a.e. on $\{x : d_M(x) > 0\}$.

Theorem 1. *The space L^M admits the introduction of a normal F-norm if and only if M is a genfunction. In particular, if M is a genfunction, then formula (1) defines in L^M a normal F-norm, called the Musielak-Orlicz F-norm.*

Let us note that $\|\varphi\|$ in (1) is meaningful for every $\varphi \in S$, but if $\varphi \in S \setminus L^M$, then $\|\varphi\|$ may be infinite. It is not difficult to see that the F-norm (1) has the property of monotonicity (if a.e. $|\varphi(x)| \leq |\psi(x)|$, then $\|\varphi\| \leq \|\psi\|$) and of **left monotone continuity** (if a.e. $|\varphi_n(x)| \uparrow |\varphi(x)|$, then $\|\varphi_n\| \uparrow \|\varphi\|$). Further, for any normal F-norm in L^M, convergence in the F-norm implies convergence in measure on every subset of finite measure (but not necessarily in measure on all of X, as is the case when the genfunction M is a function only of u). With the aid of this property one proves

Theorem 2. Let M be a genfunction. Then the space L^M, with respect to every normal F-norm, is complete, i.e., is an F-space, and any two normal F-norms in L^M are topologically equivalent.

Theorem 3. If M is a genfunction, then, for any normal F-norm, the subspace L^f_M is closed and coincides with the set of elements of the space L^M having absolutely continuous F-norms.

We give some examples of spaces L^M.

1. If $M(u, x) = u^p\rho(x)$, where $0 < p < \infty$, $\rho(x)$ is a positive measurable function, then L^M is the space L^p with weight $\rho(x)$.

2. Let $M(u, x) = \varphi(u)$, where the function φ is continuous, nondecreasing, vanishes only at zero, and $\varphi(u) \to \infty$ as $u \to \infty$. Then L^M is the generalized Orlicz space ([9]).

3. If a.e. $0 < d_M(x) < \infty$ and $M(u, x) = 0$ for $0 \leq u \leq d_M(x)$, then

   $$L^M = \{\varphi \in S_f : \operatorname{vrai\,sup} |\varphi(x)| \cdot (d_M(x))^{-1} < \infty\}.$$

   Moreover, $\|\varphi_n\| \to 0$ if and only if

   $$|\varphi_n(x)| \cdot (d_M x)^{-1} \to 0$$

   uniformly a.e.

4. If M is a pregenfunction for which the function $M(\infty, x)$ is summable on X, then $L^M = S_f$; if, in addition, M is a genfunction, then convergence

in L^M with respect to the normal F-norm is equivalent to convergence in measure on every subset of finite measure.

3°. If the pregenfunction M is a function only of u and $d_M > 0$ ($d_M = \infty$), then every measurable function bounded on X, distinct from zero on a set of finite measure, is contained in L^M (respectively in L^f_M). If, however, M is a function of u and x, then this fact, generally speaking, does not hold. For example, if $X = (0, 1)$, μ is Lebesgue measure, $M(u, x) = ux^{-1}$, then nonzero constants do not belong to $L^M = P_M$. However, every measurable bounded function that vanishes in a neighborhood of zero belongs to this space. It turns out that, also in the general case, as follows from what follows, there exist classes of bounded functions contained in L^M and L^f_M.

We shall call a chain (π) a nondecreasing sequence of measurable sets $\pi_n \subset X$ for which $\mu\pi_n < \infty$, $n = 1, 2, \ldots$.

Lemma 1 (cf. ([8])). Let $\mathfrak{A}$ be a nonempty family of measurable subsets of a measurable set $Y \subset X$, and suppose the following conditions are fulfilled:
1) if $E_1, E_2 \in \mathfrak{A}$, then $E_1 \cup E_2 \in \mathfrak{A}$;
2) if $E \subset Y$ and $\mu E > 0$, then there exists an $F \subset E$ such that $F \in \mathfrak{A}$ and $\mu F > 0$.
Then there exists a chain (π) such that all $\pi_n \in \mathfrak{A}$ and

$$\mu(Y \setminus \lim \pi_n) = 0.$$

For a given chain (π), we shall call a function φ measurable on X (π)-bounded if

$$\operatorname{vrai\,sup} |\varphi(x)| < \infty$$

and $\varphi(x) = 0$ a.e. on $X \setminus \pi_n$, beginning with some n. With the help of Lemma 1 one proves

Theorem 4. If M is a pregenfunction, then there exists a chain (π) such that

$$\lim \pi_n = \{x : d_M(x) > 0\} \quad (\lim \pi_n = \{x : d_M(x) = \infty\}),$$

and all (π)-bounded functions are contained in L^M (respectively in L^f_M).

Let now M be a genfunction, and let L^π_M be the closure of the set of all (π)-bounded functions contained in L^M, with respect to any normal F-norm. Obviously, L^π_M is a subspace of L^M. It is not difficult to show that if

$$\mu(\{x : d_M(x) = \infty\} \setminus \lim \pi_n) = 0,$$

then $L^f_M \subset L^\pi_M$. Moreover, as follows from Theorem 4, there always exists a chain (π) such that

$$\lim \pi_n = \{x : d_M(x) = \infty\}$$

and $L^f_M = L^\pi_M$. Relying on this assertion, one can show that if the measure μ has a countable basis ([1]), then L^f_M is separable.

4°. Here we shall consider some metric properties of the space L^M endowed with the F-norm (1). First of all, note that if $\varphi \in P_M$, then

$$\|\varphi\| \leq \|d_M\|,$$

as follows from the monotonicity of the F-norm (1).

Lemma 2. If M is a pregenfunction, then there exists a nondecreasing sequence of nonnegative functions $\varphi_n \in P_M$ such that

$$\lim \varphi_n(x) = d_M(x) \quad \text{a.e.}$$

Theorem 5. If M is a genfunction, then

$$\|d_M\| = \sup\{\|\varphi\| : \varphi \in P_M\}.$$

Remark 1. From Lemma 2 there also follows the following proposition, more general than Theorem 5. Let f be a functional on S with values

on $[0, \infty]$, possessing the property of monotone left-continuity. Then, if M is a pregenfunction, then

$$f(d_M) = \sup\{f(\varphi) : \varphi \in P_M\}.$$

A natural supplement to Theorem 5 is

Theorem 6. Let M be a genfunction. Then

$$\{\varphi \in L^M : \|\varphi\| \leq 1\} \subset \{\varphi \in L^M : I_M\varphi \leq 1\} \subset P_M;$$

moreover,

$$P_M = \{\varphi \in L^M : \|\varphi\| \leq 1\}$$

if and only if $I_M(d_M) \leq 1$.

Let us now consider the question of the position of the set P_M relative to the subspace L^f_M. To this end put

$$\rho(\varphi, L^f_M) = \inf\{\|\varphi - \psi\| : \psi \in L^f_M\}.$$

Theorem 7. Let M be a genfunction. Then

$$\{\varphi \in L^M : \rho(\varphi, L^f_M) < 1\} \subset P_M.$$

Moreover, if $\mu\{x : 0 < d_M(x) < \infty\} = 0$, then

$$P_M \subset \{\varphi \in L^M : \rho(\varphi, L^f_M) \leq 1\}.$$

5°. Let the genfunction M be convex in u for almost every x. Then $M(\infty, x) = \infty$ a.e. and the function M is continuous in u on $[0, d_M(x))$ for almost every x for which $d_M(x) > 0$. Such a genfunction is called a Young function, and the space L^M generated by the Young function will be called (5) an Orlicz-Nakano space, since this space was first described in (12), and a particular case of it (when M is a function only of u) is an Orlicz space, more precisely an Orlicz space in the sense of Zaanen (14, 15) (cf. (2, 3)).

The Orlicz-Nakano space is a Banach space with norm

$$\|\varphi\|_1 = \inf\{\varepsilon > 0 : I_M(\varepsilon^{-1}\varphi) \leq 1\}$$

or

$$\|\varphi\|_2 = \inf\{\alpha^{-1}(1 + I_M(\alpha\varphi)) : 0 < \alpha < \infty\}$$

(for $\|\varphi\|_2$ there is also another expression (2, 5), using the complementary Young function). Both norms are normal and, consequently, topologically equivalent to the F-norm (1) (see also (10), where inequalities are established for the F-norm (1) and the norm $\|\cdot\|_1$). We also note that, as in Orlicz spaces,

$$\|\varphi\|_1 \leq \|\varphi\|_2 \leq 2\|\varphi\|_1.$$

The basic facts of the theory of Orlicz spaces extend to Orlicz-Nakano spaces. Here we shall give several propositions generalizing some results from (4, 6).

Let M be a Young function;

$$\rho_k(\varphi, L_M^f) = \inf\{\|\varphi - \psi\|_k : \psi \in L_M^f\},$$

$$\Pi_k = \{\varphi \in L^M : \rho_k(\varphi, L_M^f) < 1\}, \quad k = 1, 2.$$

It is easy to see that

$$\overline{\Pi}_k = \{\varphi \in L^M : \rho_k(\varphi, L_M^f) \leq 1\}, \quad k = 1, 2.$$

Theorem 8. $\Pi_2 \subset \Pi_1 \subset P_M$.

Put

$$X^0 = \{x : 0 < d_M(x) < \infty\}.$$

Theorem 9. If $\mu X^0 = 0$, then

$$\Pi_1 = \Pi_2 = \operatorname{int} P_M, \qquad \overline{\Pi}_1 = \overline{\Pi}_2 = \overline{P}_M.$$

Corollary. If $\mu X^0 = 0$, then

$$\rho_1(\varphi, L_M^f) = \rho_2(\varphi, L_M^f) = \inf\{\alpha > 0 : \alpha^{-1}\varphi \in P_M\}$$

for every $\varphi \in L^M$, i.e. the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ generate one and the same norm in the quotient space L^M/L_M^f (cf. (7), p. 8).

Theorem 10. If $\mu X^0 > 0$, then

$$\max\{r : D_r \subset P_M\} = 1,$$

where

$$D_r = \{\varphi : \|\varphi\|_1 \leq r\}.$$

Moreover, if $I_M(d_M) \leq 1$, then $\|d_M\|_1 = 1$ and $P_M = D_1$; if $I_M(d_M) > 1$, then $\|d_M\|_1 > 1$ and both inclusions

$$D_1 \subset P_M \subset \{\varphi : \|\varphi\|_1 \leq \|d_M\|_1\}$$

are proper.

Remark 2. Since the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ have the property of monotone left-continuity, by Remark 1,

$$\|d_M\|_k = \sup\{\|\varphi\|_k : \varphi \in P_M\}, \quad k = 1, 2.$$

It follows that the set P_M is bounded in the Orlicz-Nakano space L_M if and only if $d_M \in L^M$.

In conclusion we note that a number of properties of the spaces L_M (conditions for the embedding of one space in another, criteria for closedness and openness of the set P_M, etc.) can be obtained in the form of simple consequences of certain propositions on the Nemytskii operator, whose investigation in spaces generated by genfunctions is carried out in (13).

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Proof correction note. After the manuscript had been submitted for publication, the author became acquainted with the work [(16)], in which spaces generated by genfunctions are considered under the condition that $d_M(x) > 0$ for all x.

Tambov Institute
of Chemical Machine-Building

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* *Correction.* In article [(13)] the following corrections must be made:

p. 64, formula (1) should read

$$\Phi[\beta|g(u,x)|,x] \leq \gamma M[\alpha|u|,x] + f(x). \tag{1}$$

p. 64, line 25 from the bottom: printed $h[P_M^\alpha(\Delta)L^\Phi$, should read $h[P_M^\alpha(\Delta)] \subset L^\Phi$.

p. 64, line 21 from the bottom: printed " ," should read " ."

p. 66, line 4, should read

$$\Phi[\beta|g(u'',x) - g(u',x)|] \leq \gamma[M(|u'|,x) + M(\alpha|u'' - u'|,x)] + f(x).$$

p. 66, line 5: printed α, should read a.

Note: Figure translations are in progress. See original paper for figures.

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