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Abstract

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ONE-DIMENSIONAL NONSTATIONARY FLOWS OF A BAROTROPIC FLUID

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Nonlinear motions of a rarefied plasma can, in a number of cases, be described exactly by the Euler equations for a barotropic fluid with adiabatic index $\gamma = 2$. For example, the motion of a plasma across a magnetic field is described by the equations of Chu, Goldberger, and Low ([1]), which have the form of the Euler equations with $\gamma = 2$. Analogous equations describe plasma motions in a high-frequency electromagnetic field ([2]) and the theory of "shallow water" ([4]). Let us note that the same equations describe the propagation of narrow plane beams in nonlinear geometrical optics (for a defocusing medium $\partial\varepsilon/\partial y|_{y=0} < 0$) ([3]). At the same time, flows of a barotropic fluid for $\gamma = 2$ possess a number of interesting features, to the study of which the present work is devoted.

1. Conservation laws. The Euler equations for nonstationary one-dimensional flows of a barotropic fluid for $\gamma = 2$ ($P = {}^1/_2\beta\rho^2$) have the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0; \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \beta\frac{\partial\rho}{\partial x} = 0. \tag{1}$$

These equations can be represented in the form of conservation laws:

$$\partial P/\partial t = -\partial Q/\partial x. \tag{2}$$

It follows from (2) that, for any value of t, the integral

$$I = \int_{-\infty}^{\infty} P\,dx \tag{3}$$

is constant (if, of course, the function P decreases faster than x^{-1} as $x \to \infty$; $Q(+\infty) = Q(-\infty)$).

Equations (1), written in the form (2), express the conservation laws of mass and the velocity integral. Combining equations (1), it is not difficult to obtain from them, in the usual way, the conservation laws of momentum and energy. In these cases, respectively,

$$P_1 = \rho v; \qquad Q_1 = \rho v^2 + {}^1/{}_2\beta\rho^2;$$

$$P_2 = {}^1/{}_2\rho v^2 + {}^1/{}_2\beta\rho^2; \qquad Q_2 = {}^1/{}_2\rho v^3 + \beta v\rho^2. \tag{4}$$

The quantity Q_1 is the single component of the momentum-flux-density tensor different from 0, P_2 is the energy density of the fluid, and Q_2 is the energy-flux-density vector.

An essential feature of the equations of a barotropic fluid with $\gamma = 2$ is that they allow one to find other conservation laws as well. Indeed, considering polynomials P_n, homogeneous in powers of ρ and v^2, ne-

it is difficult for each polynomial to construct, with the aid of equations (1), relations of type (2). For example,

$$\begin{aligned}
P_3 &= \rho v^3 + 3\beta v\rho^2; & Q_3 &= \rho v^4 + \frac{9}{2}\beta\rho^2 v^2 + \beta^2\rho^3;\\
P_4 &= \rho v^4 + 6\beta\rho^2 v^2 + 2\beta^2\rho^3; & Q_4 &= \rho v^5 + 8\beta\rho^2 v^3 + 6\beta^2\rho^3 v;\\
P_5 &= \rho v^5 + 10\beta\rho^2 v^3 + 10\beta^2 v\rho^3; & Q_5 &= \rho v^6 + \frac{25}{2}\beta\rho^2 v^4 + 20\beta^2 v^2\rho^3 + \frac{5}{2}\beta^3\rho^4;\\
P_6 &= \rho v^6 + 15\beta\rho^2 v^4 + 30\beta^2\rho^3 v^2 + 5\beta^3\rho^4; & Q_6 &= \rho v^7 + 18\beta\rho^2 v^5 + 50\beta^2\rho^3 v^3 + 20\beta^3\rho^4 v;
\end{aligned} \tag{5}$$

and so on. This series can be continued to infinity: for each moment of the velocity v there is a definite conservation law.

It follows from this, in particular, that for an arbitrary initial distribution of density $\rho_0(x)$ and velocity $v_0(x)$, one can, according to (3)–(5), determine an infinite number of constants I_n that are conserved throughout the motion*. In particular, if at the initial instant the fluid is at rest ($v_0(x) = 0$), then

$$I_{2n+1} = 0; \qquad I_{2n} = C\beta^n \int_{-\infty}^{\infty} [\rho_0(x)]^{n+1}\, dx. \tag{6}$$

Consequently, in this case all moments of the initial density distribution $\rho_0(x)$ are conserved.

2. Exact solutions. Riemann simple waves—particular solutions of the Euler equations—play an important role in the qualitative analysis of phenomena arising in nonlinear hydrodynamics. The equations of a barotropic fluid with

$\gamma = 2$, in addition to Riemann simple waves, admit another broad class of exact nonlinear solutions. To find them, let us make a hodograph transformation in equations (1), i.e., we shall regard ρ and v as the new variables, and x and t as unknown functions. After the usual calculations (see (4)) we arrive, instead of (1), at the following linear equations for the functions $x(u,\rho)$, $t_1(u,\rho)$:

$$\partial x/\partial u - u\,\partial t_1/\partial u + \rho\,\partial t_1/\partial\rho = 0; \qquad \partial t_1/\partial u - u\,\partial t_1/\partial\rho + \partial x/\partial\rho = 0. \quad (7)$$

Here $u = v\beta^{-1/2}$, $t_1 = t\beta^{1/2}$ (the subscript 1 will be omitted below).

The equations (7) for the functions x and tu are homogeneous with respect to ρ and u^2. Therefore their particular solutions may be sought in the form of polynomials homogeneous in powers of ρ and u^2. We shall assume first that x is an even function of u:

$$x_n(u,\rho) = \sum_{k=0}^{n} a_{nk} u^{2k} \rho^{n-k}; \qquad t_n(u,\rho) = u^{-1} \sum_{k=0}^{n} b_{nk} u^{2k} \rho^{n-k}. \quad (8)$$

Substituting these expressions into equations (7) and equating the terms with equal powers $u^{2k}\rho^{n-k}$, we arrive at the recurrence relations

$$b_{nk} = \frac{k-n-1}{2k-1}\,(a_{n,k-1} - b_{n,k-1});$$

$$a_{nk} = \frac{(n+1-3k)(n-k+1)}{2k(2k-1)}\,(a_{n,k-1} - b_{n,k-1}), \quad (9)$$

which relate the coefficients a_{nk} and b_{nk}. Setting $a_{n0} = 1$, $b_{n0} = 0$, we successively find the functions x_n and t_n—particular solutions of equations (7):

$$\begin{aligned} x_1 &= \rho - \frac{1}{2}u^2; & t_1 &= -u; \\ x_2 &= \rho^2 - \frac{1}{2}u^4; & t_2 &= -2\rho u - \frac{2}{3}u^3; \\ x_3 &= \rho^3 + \frac{3}{2}\rho^2 u^2 + \frac{3}{2}\rho u^4 + \frac{1}{6}u^6; & t_3 &= -3u\rho^2 - 3\rho u^3 - \frac{3}{10}u^5. \end{aligned} \quad (10)$$

and so on.

Each of the solutions found corresponds to definite initial conditions for equation (1):

$$\rho = \rho_0(x), \qquad u = u_0(x) \quad \text{at } t = 0. \quad (11)$$

* This assertion is valid only in the absence of strong discontinuities, since on the lines of strong discontinuities the conservation laws (5) may fail to hold.

Let us find them. We take into account that conditions (11) define in the space (ρ, u), in parametric form (with parameter x), a certain curve $\rho(u)$. On this curve $t = 0$, and $x = x_0(\rho)$, where the function $x_0(\rho)$ is the inverse of $\rho_0(x)$. In our case the condition $t = 0$ is satisfied for all solutions when $u = 0$. Thus, each of the solutions found, x_n, t_n, describes a one-dimensional motion of the fluid under the initial conditions: $u_0(x) = 0,\ \rho_0(x) = x^{1/n}$.

It is not difficult to verify that each of the solutions (10), x_n, t_n, is self-similar: it can be represented in the form

$$\rho = t^{2/(2n-1)} f_{1n}(\tau); \qquad u = t^{1/(2n-1)} f_{2n}(\tau); \qquad \tau = x t^{-2n/(2n-1)}. \tag{12}$$

It is essential that equations (7) for the functions $x(u, \rho),\ t(u, \rho)$ are linear. This means that not only the functions x_n, t_n, but also an arbitrary linear combination of them

$$x = \sum_{n=1}^{\infty} C_n x_n(u, \rho); \qquad t = \sum_{n=1}^{\infty} C_n t_n(x, \rho) \tag{13}$$

satisfies equations (7). The obtained class of solutions is characterized by an infinite sequence of constants $C_1, C_2, \dots, C_n$, or else by an arbitrary function expandable in a series. Indeed, suppose, for example, that at the initial moment the fluid was at rest, i.e., at $t = 0$,

$$u(x, 0) = 0; \qquad \rho(x, 0) = \rho_0(x). \tag{14}$$

According to (11), when $u = 0,\ t_n = 0,\ x_n = \rho^n$. Therefore, expanding in a Taylor series the function $x_0(\rho)$, inverse to $\rho_0(x)$,

$$x_0(\rho) = x_0 + \sum_{n=1}^{\infty} C_n \rho^n; \qquad C_n = \frac{1}{n!}\left[\frac{\partial^n x_0(\rho)}{\partial \rho^n}\right]_{\rho=0}, \tag{15}$$

we determine the constants C_n. In other words, formulas (13) with coefficients C_n determined according to (15) give, in implicit form, the solution of equations (1) with boundary conditions (14). Of course, the solution (13) is no longer self-similar. The development of the solution (13) leads to a gradual steepening of the functions $\rho(x)$ and $u(x)$, accompanied at some moment t by the occurrence of singular points, which is entirely analogous to the behavior of a simple Riemann wave. The values x_k and t_k corresponding to the appearance of a singular point are found from the system of equations [4] $\partial x/\partial \rho|_t = 0;\ \partial^2 x/\partial \rho^2|_t = 0$. Near a singular point, $\rho - \rho_k \sim (x - x_k)^{1/3};\ u - u_k \sim (x - x_k)^{1/3}$.

We have considered the case when x is an even function of u. Analogous solutions can also be found in the case when x is an odd function of u. They have the form:

$$\begin{aligned}
x_1 &= -u\rho + \frac{2}{3}u^3, & t_1 &= \rho + u^2; \\
x_2 &= -u\rho^2 + \frac{1}{2}\rho u^3 + \frac{1}{5}u^5, & t_2 &= \frac{1}{2}\rho^2 + \frac{3}{2}\rho u^2 + \frac{1}{4}u^4; \\
x_3 &= -u\rho^3 + \frac{3}{5}\rho u^5 + \frac{2}{35}u^7, & t_3 &= \frac{1}{3}\rho^3 + 2\rho^2 u^2 + \rho u^4 + \frac{1}{15}u^6
\end{aligned}$$

and so on. These solutions also represent self-similar functions $\rho(x,t)$ and $u(x,t)$: $\rho = t^{1/n} f_{1n}(\tau),\ u = t^{1/2n} f_{2n}(\tau),\ \tau = xt^{-(2n+1)/2n}$.

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Note: Figure translations are in progress. See original paper for figures.