ON THE COMPLETENESS OF A SYSTEM OF ANALYTIC FUNCTIONS

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Abstract

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Let a system of functions $\{\varphi_n(z)\}$, regular in a domain D, be complete in this domain*. We note that from the definition of a complete system the following criterion for the completeness of a system of regular functions follows.

In order that a system of functions $\{\psi_n(z)\}$ be complete in a domain D, it is necessary and sufficient that for each function $\varphi_m(z)$ there exist a sequence of linear combinations of the functions $\psi_n(z)$ converging uniformly to the function $\varphi_m(z)$ $(m = 1, 2, ...)$ in the domain D. In particular, if D is a simply connected domain not containing the point at infinity, then as the system $\{\varphi_n(z)\}$ in this criterion one may take the system $\{z^n\}$ $(n = 0, 1, ...)$ (see ([3]) or ([6])).

We record one consequence of this criterion.

If a system of functions $\{\varphi_n(z)\}$, regular in a simply connected domain D, is complete in this domain, then it is also complete in any simply connected subdomain of the domain D; in particular, it is complete in each component of the intersection of the domain D with any circle.

We shall need one more definition. Let $A(D)$ be the set of all functions regular in the domain D. A system of functions $\{\varphi_n(z)\}$, regular in a domain $D_* \subset D$, is called **complete in the class** $A(D)$ **on the domain** D_*, if for any function $f(z) \in A(D)$ there exists a sequence of linear combinations of the functions $\{\varphi_n(z)\}$ converging uniformly to the function $f(z)$ in the domain D_*. From what was said above it follows that a system of functions $\{\varphi_n(z)\}$, complete in the class $A(D)$ on the domain $D_* \subset D$, is complete in the domain D_*, if the domains D and D_* are simply connected.

We shall prove two theorems on the completeness of a system of analytic functions of the form $\{F(z, a_n)\}$, where $\{a_n\}$ is a sequence of complex numbers.

Theorem 1. *Let a system of functions $\{\varphi_n(z)\}$, analytic in the finite disk $|z| \le R$, be complete in this disk, and let the series*

$$F(z, u) = \sum_{n=0}^{\infty} \varphi_n(z) u^n \tag{1}$$

converge uniformly in z in the disk $|z| < R$ and in u for $|u| \leq 1$. Then the system of functions $\{F(z, a_n)\}$ is complete in the disk $|z| < R$ for any set of points $\{a_k\}$, where $|a_k| \leq 1$ $(k = 0, 1, ...)$, provided only that the series $\sum_{k=0}^{\infty}(1-|a_k|)$ diverges.

Proof. Representing the function $\partial^m F(z, u)/\partial u^m$ by the Cauchy integral with respect to u for fixed z $(|z| \leq R)$ and then putting $u = 0$, we find:

$$\left[\frac{\partial^m F(z, u)}{\partial u^m}\right]_{u=0} = m!\varphi_m(z) = \frac{m!}{2\pi i}\int_{|\xi|=1} \frac{F(z, \xi)}{\xi^{m+1}}\, d\xi. \tag{2}$$

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* Whatever regular function $f(z)$ in D may be, there exists a sequence of linear combinations

$$\sum_{n=1}^{p_k} c_{n,k}\varphi_n(z)$$

with constant coefficients, which converges uniformly in the domain D to the function $f(z)$ (see, for example, ([7]), p. 274).

It is known that for $|\xi| = 1$ there exists a sequence of numbers $\{A_k\}$ such that

$$\frac{1}{\xi^{m+1}} = \sum_{k=0}^{n} \frac{A_k}{\xi - \alpha_k} + \varepsilon_n(\xi),$$

if the series $\sum(1 - |\alpha_k|)$ diverges, where $|\varepsilon_n(\xi)| \to 0$ as $n \to \infty$ $(|\xi| = 1)$ (see ([3]), p. 48). Thanks to this, from equality (2) we find:

$$\varphi_m(z) = \sum_{k=0}^{n} A_k F(z, \alpha_k) + \delta_n(z),$$

where the series

$$\sum_{k=0}^{\infty}(1 - |\alpha_k|)$$

diverges and $|\delta_n(z)| \to 0$ as $n \to \infty$ $(|z| \leq R)$. Hence, by the completeness criterion for systems of analytic functions, it follows that the system $\{F(z, \alpha_k)\}$ is complete in the disk $|z| < R$ if and only if the series $\sum(1 - |\alpha_k|)$ diverges.

Corollary 1. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad c_n \neq 0 \quad (n \neq 0, 1, ...),$$

be an analytic function in the finite disk $|z| \leq R$. Then the system $\{f(\alpha_k z)\}$ is complete in the same disk $|z| < R$ for any set of points $\{\alpha_k\}$, where $|\alpha_k| \leq 1$ $(k = 0, 1, ...)$, if the series $\sum(1 - |\alpha_n|)$ diverges.*

Indeed, assuming that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad c_n \neq 0 \quad (n = 0, 1, ...),$$

is an analytic function in the disk $|z| \leq R$ and choosing $\varphi_n(z) = c_n z^n$, we find

$$F(z, u) = \sum_{n=0}^{\infty} c_n (zu)^n = f(zu) \qquad (|u| \leq 1).$$

Theorem 2. Let the system of functions $\{\varphi_n(z)\}$, analytic in a simply connected domain D, be complete in this domain; let the function $F(z, u)$ be defined by the series (1), converging uniformly in z in the domain D for any u; let $n(r)$ be the density function of the sequence $\{\alpha_n\}$; and, moreover,

$$M(F; D_*; r) = \max_{z \in D_*,\, |u| \leq r} |F(z, u)| \qquad (D_* \subset D).$$

Then the system of functions $\{F(z, \alpha_n)\}$ is complete in the class $A(D)$ on the domain D_*, if in the domain D_* the inequality

$$\ln M(F; D_*; r/\theta) < c(\theta) n(r), \tag{3}$$

holds, where $c(\theta) < \ln 1/\theta$ and $\theta (0 < \theta < 1)$ is a fixed number.

Proof. In ([3]) it is shown that the function

$$\Phi_n(t, u) = \prod_{k=1}^{n} \frac{t - \alpha_k}{u - \alpha_k} \frac{1}{|\alpha_n|^{2n}(u - t)} \prod_{k=1}^{n} \left[|\alpha_n|^2 - \theta^2 \overline{\alpha_k}(u - t)\right]$$

can be represented in the form

$$\Phi_n(t, u) = \frac{1}{u - t} - \sum_{k=1}^{n} \frac{q_{n,k}(t)}{u^{-\alpha_k}}, \tag{4}$$

where

$$q_{n,k}(t) = \prod_{\substack{j=1 \\ (j \neq k)}}^{n} \frac{t - \alpha_j}{\alpha_k - \alpha_j} \prod_{j=1}^{n} \left[1 - \frac{\theta^2 \overline{\alpha_k}(\alpha_j - t)}{|\alpha_n|^2}\right].$$

* Proved by the author in [3].

Multiplying equality (4) by $\frac{1}{2\pi i}F(z,u)$ and integrating along the circle $\theta|u| = |\alpha_n|$ $(0 < \theta < 1)$, we obtain

$$F(z,t) - \sum_{k=1}^{n} q_{n,k}(t)F(z,\alpha_k) = \frac{1}{2\pi i}\int_{\theta|u|=|\alpha_n|} \Phi_n(t,u)F(z,u)\,du.$$

We differentiate this equality m times with respect to t, divide the result obtained by $m!$, and then set $t = 0$. Then, putting $c_{k,m} = \frac{1}{m!}q_{n,k}^{(m)}(0)$, we obtain:

$$\left|\varphi_m(z) - \sum_{k=1}^{n} c_{k,m}F(z,\alpha_k)\right| = |R_{n,m}(z)| \leq$$

$$\leq \frac{|\alpha_n|}{\theta\rho^m} \max_{\theta|u|=|\alpha_n|,\ |\xi|=\rho} |\Phi(\xi,u)| \max_{\theta|u|=|\alpha_n|,\ z\in D} |F(z,u)|. \tag{5}$$

Using the estimate obtained in paper [3] for $\max|\Phi(\xi,u)|$, it is not difficult to show that

$$|R_{n,m}(z)| \leq B\exp\left\{-n(r)\left[\ln\frac{1}{\theta} - \frac{\ln M(F;D_*;r/\theta)}{n(r)}\right]\right\}, \tag{6}$$

where B is a constant independent of r. By the above-mentioned completeness criterion for systems of analytic functions, as inequalities (5) and (6) show, the system of functions $\{F(z,\alpha_n)\}$ is complete in the domain D_* in the class $A(D)$ when condition (3) is satisfied.

Corollary 1. Let the function $\varphi_n(z)$ in the domain D satisfy the condition

$$\lim_{n\to\infty} n^{1/\rho}|\varphi_n(z)|^{1/n} = (\sigma e\rho)^{1/\rho}|z| \qquad (z \in D) \tag{7}$$

uniformly in z in the domain D, where ρ and σ are some positive numbers, the function $F(z,u)$ is defined by equality (1), and the sequence of numbers $\{\alpha_n\}$ is such that

$$\overline{\lim_{n\to\infty}} \frac{n}{|\alpha_n|^\mu} = \nu < \infty.$$

Then*: 1) in the case $\mu > \rho$, the system of functions $\{F(z,\alpha_n)\}$ is complete in the class $A(D)$ in every domain $D_* \subset D$; 2) in the case $\mu = \rho$, the system

$\{F(z, \alpha_n)\}$ is complete in the class $A(D)$ on each component of the intersection of the domain D with the disk

$$|z| < (\nu/\rho e \sigma)^{1/\rho}. \tag{8}$$

Proof. By condition (7), the series (1) converges uniformly for all $z \in D$ and arbitrary u, and, moreover, we have

$$M(F; D_*; r) \leq K(\varepsilon) \exp[(\sigma + \varepsilon)(|z|r)^\rho],$$

where $z \in D_* \subset D$ and $\varepsilon > 0$.

Choosing $r = (n/\nu)^{1/\mu}$ and $\theta = e^{-1/\rho}$ and observing that $n(r) = n$, from inequality (3) we obtain

$$\sigma|z|^\rho (n/\nu)^{\rho/\mu} < n/\rho e \qquad (z \in D_* \subset D).$$

Hence it is seen that in the case $\mu > \rho$ the system $\{F(z, \alpha_n)\}$ is complete in the class $A(D)$ in every domain $D_* \subset D$, while in the case $\mu = \rho$ the system $\{F(z, \alpha_n)\}$ is complete in the class $A(D)$ on each component of the intersection of the domain with the disk (8).

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* The first part was proved by another method by A. A. Mirolyubov ([4]), while the second part is a refinement of his corresponding theorem.

Corollary 2. Let $M(r)$ be the maximum modulus of the entire function

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \quad (c_k \neq 0,\ k = 0, 1, ...)$$

in the disk $|z| \leq r$; let $n(r)$ be the density function of the sequence $\{\alpha_n\}$. The system $\{f(\alpha_n z)\}$ is complete in the disk $|z| < R$, if the inequality

$$\ln M(Rr/\theta) < c(\theta) n(r), \tag{9}$$

holds, where $c(\theta) < \ln 1/\theta$ and θ $(0 < \theta < 1)$ is a fixed number.

Indeed, putting $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and $\varphi_n(z) = c_n z^n$, from equality (1) we find

$$F(z, u) = \sum_{n=0}^{\infty} c_n (zu)^n = f(zu).$$

In this case the domain D is the whole plane and the class $A(D)$ consists of functions regular in a neighborhood of $z = 0$,

$$M(F;\ |z| \leq R;\ r) = \max_{|z|\leq R,\ |u|<r} |f(zu)| = M(Rr),$$

and inequality (3) is written in the form (9).

This assertion was proved in the paper ([3]), and it was shown there that the corresponding theorems of A. O. Gelfond ([1]) and A. I. Markushevich ([2]) on the completeness of the system $\{f(\alpha_n z)\}$ under various assumptions concerning the nature of $f(z)$ and the sequence $\{\alpha_n\}$ follow from it.

From Theorem 2 there also follows the following assertion (see ([4]), p. 282).

Corollary 3. If the entire function

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

satisfies the condition $c_k \neq 0$ $(k = 0, 1, \ldots)$ and has type not exceeding σ for the refined order $\rho(r)$, and $\{\alpha_k\}$ is a sequence of complex numbers, then the system of functions $\{f(\alpha_k z)\}$ is complete in the disk with center at zero and radius R, determined by the equality (see ([7]), p. 283)

$$R^\rho = \frac{1}{e\rho\sigma} \lim_{n\to\infty} \frac{n(|\alpha_n|)}{|\alpha_n|^{\rho(|\alpha_n|)}}.$$

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Note: Figure translations are in progress. See original paper for figures.