UNIFORM SPACES AND PERFECT IRREDUCIBLE MAPPINGS OF TOPOLOGICAL SPACES

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MATHEMATICS

V. V. FEDORCHUK

UNIFORM SPACES AND PERFECT IRREDUCIBLE MAPPINGS OF TOPOLOGICAL SPACES

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This note introduces the concept of a θ-uniform space. This concept generalizes the notion of a uniform space compatible with a given topological space. θ-uniformities exist on every Hausdorff space. Just as every uniformity naturally generates a proximity, every θ-uniformity generates a θ-proximity.*

As in the case of θ-proximities, there is a one-to-one correspondence between θ-uniformities on a given topological space and uniformities on its completely regular preimages with respect to θ-perfect irreducible mappings. In this situation ordinary uniformities are distinguished among all θ-uniformities as projective objects with respect to the class of uniform regular θ-mappings.

§ 1. We shall say that a system $\nu = \{V\}$ of canonically open subsets of a topological space X is a θ-covering of the space X of **locally finite type** if, for every point $x \in X$, there exists a finite collection $V_1, \dots, V_n$ of elements of the system ν such that

$$x \in \left\langle \bigcup_{i=1}^{n} [V_i] \right\rangle .$$

A large stock of θ-coverings of locally finite type is provided by

Lemma 1. *Let $f : Y \to X$ be a θ-perfect irreducible mapping and let $\nu = \{V\}$ be a covering of the space Y by canonically open sets. Then the system $f^{\#}\nu = \{f^{\#}V \mid V \in \nu\}$ will be a θ-covering of locally finite type of the space X.*

A family $\mathfrak{B} = \{\nu\}$ of θ-coverings of locally finite type of a topological space X is called a θ-**uniformity** if the following axioms are satisfied:

$\mathrm{I_U}$. If a θ-covering $\nu \in \mathfrak{B}$ is inscribed in a θ-covering w of locally finite type, then $w \in \mathfrak{B}$.

II_U. For any two θ-coverings $u, \nu \in \mathfrak{B}$ there exists a θ-covering $w \in \mathfrak{B}$ which is star-inscribed both in the θ-covering u and in the θ-covering ν.

III_U. If x, y are distinct points of the space X, then there exist neighborhoods G and H of these points and a θ-covering $\nu \in \mathfrak{B}$ such that

$$H \cap \mathrm{st}_\nu G = \emptyset.$$

IV_U. For every point $x \in X$ and for each of its canonically open neighborhoods G there exists a neighborhood H of the point x and a θ-covering $\nu \in \mathfrak{B}$ such that $\mathrm{st}_\nu H \subset G$. Thus, if in the axioms of θ-uniformity the θ-coverings of locally finite type are replaced by coverings, one obtains one of the variants of the axioms of a separated uniformity compatible with the given topology.

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* θ-proximities are studied in detail in work ([1]), whose terminology and notation we use here as well.

A topological space X endowed with a θ-uniformity $\mathfrak{B}$ is called a θ-**uniform space** $(X, \mathfrak{B})$. A **fundamental system** or **base** of a θ-uniformity is any set $\mathfrak{B}_1 \subset \mathfrak{B}$ having the property that into every θ-cover $v \in \mathfrak{B}$ one can inscribe a θ-cover $v_1 \in \mathfrak{B}_1$. From a fundamental system the θ-uniformity is uniquely recovered by virtue of axiom I_U.

Every separated uniformity is a θ-uniformity. On an extremally disconnected space every θ-uniformity is a uniformity, since every locally finite θ-cover of an extremally disconnected space is a cover. An example of a θ-uniformity on an arbitrary Hausdorff space X is the θ-uniformity whose base is the system of all finite θ-covers by canonically open sets.

Every θ-uniformity $\mathfrak{B}$ on a space X generates a θ-proximity: $A\theta B \iff$ there exist neighborhoods C and D of the sets A and B, respectively, and a θ-cover $v \in \mathfrak{B}$ such that $C \cap \mathrm{st}_v D = \emptyset$. We shall denote this θ-proximity by the symbol $\theta_{\mathfrak{B}}$.

The most general example of a θ-uniformity is given by:

Theorem 1. *Let $f : Y \to X$ be a θ-perfect irreducible mapping of a completely regular space Y onto a space X. Let a uniformity $\mathfrak{B}$ be given on the space Y. Then the family*

$$f^\# \mathfrak{B} = \{\, f^\# v \mid v \in \mathfrak{B},\ v \text{ consists of canonically open sets} \,\}$$

is a θ-uniformity on the space X. Moreover,

$$\theta_{f^\# \mathfrak{B}} = f\theta_{\mathfrak{B}}{}^*.$$

The following theorem shows that there exists only one uniformity generating a given θ-uniformity.

Theorem 2. *Let $f_i : Y_i \to X$ be θ-perfect irreducible mappings of completely regular spaces onto the space X, $i = 1, 2$. Let uniformities $\mathfrak{B}_i$ be given on the spaces Y_i such that*

$$f_1^{\#}\mathfrak{B}_1 = f_2^{\#}\mathfrak{B}_2.$$

Then there exists a uniformly continuous in both directions homeomorphism $h : Y_1 \to Y_2$ such that $f_1 = f_2 h$.

Theorem 3 asserts that every θ-uniformity is generated by some uniformity, unique by Theorem 2.

Theorem 3. *Let $(X, \mathfrak{B})$ be a θ-uniform space. There exists a uniform space $(X_{\mathfrak{B}}, \widetilde{\mathfrak{B}})$ and a θ-perfect irreducible mapping $\pi_{\mathfrak{B}} : X_{\mathfrak{B}} \to X$ such that*

$$\mathfrak{B} = \pi_{\mathfrak{B}}^{\#}\widetilde{\mathfrak{B}}.$$

§ 2. A mapping $f : (X, \mathfrak{B}) \to (Y, \mathfrak{M})$ is called **θ-uniformly continuous** if: 1) for every θ-cover $w \in \mathfrak{M}$ there exists a θ-cover $v \in \mathfrak{B}$ such that the cover $\{f[V] \mid V \in v\}$ is inscribed in the cover $\{W \mid W \in \mathfrak{M}\}$; 2) the family $\{\langle[f^{-1}W]\rangle \mid W \in w\}$ is a uniform θ-cover of the space $(X, \mathfrak{B})$.

Lemma 2. *If $f : (X, \mathfrak{B}) \to (Y, \mathfrak{M})$ is a θ-uniformly continuous mapping, then the mapping*

$$f : (X, \theta_{\mathfrak{B}}) \to (Y, \theta_{\mathfrak{M}})$$

is θ-proximally continuous.

The following theorem is analogous to Theorem 4 of [1] and is proved with its aid.

Theorem 4. *Let $f : (X, \mathfrak{B}) \to (Y, \mathfrak{M})$ be a θ-uniformly continuous mapping onto a regular space Y. Then there exists a uniformly continuous mapping*

$$\tilde{f} : (X_{\mathfrak{B}}, \widetilde{\mathfrak{B}}) \to (Y_{\mathfrak{M}}, \widetilde{\mathfrak{M}})$$

such that the diagram is commutative

$$\begin{array}{ccc} X_{\mathfrak{B}} & \xrightarrow{\tilde{f}} & Y_{\mathfrak{M}} \\ \downarrow \pi_{\mathfrak{B}} & & \downarrow \pi_{\mathfrak{M}} \\ X & \xrightarrow{f} & Y \end{array}$$

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* By $\delta_{\mathfrak{B}}$ is denoted the proximity generated on the space Y by the uniformity $\mathfrak{B}$, and by $f\theta_{\mathfrak{B}}$ the θ-proximity generated on X by this proximity and the mapping $f : Y \to X$ (see Theorem 1 of [1]).

Remark. The requirement of regularity of the space Y in this theorem can be weakened only to the requirement of continuity of the mapping f, since there exists an example showing that Theorem 4 of ([1]), which follows from the present theorem, ceases to be true if the regularity of the space Y is omitted. The regularity of the space Y can be omitted for closed and irreducible mappings.

Theorem 5. Let $f : (X, \mathfrak{V}) \to (Y, \mathfrak{W})$ be a θ-uniformly continuous mapping of the space X onto the space Y. Suppose, moreover, that the mapping f is closed* and irreducible. Then there exists a uniformly continuous mapping

$$\tilde{f} : (X_{\mathfrak{V}}, \widetilde{\mathfrak{V}}) \to (Y_{\mathfrak{W}}, \widetilde{\mathfrak{W}})$$

such that

$$\pi_{\mathfrak{W}} \tilde{f} = f \pi_{\mathfrak{V}}.$$

If, in addition, the mapping f is bicompact, then the mapping $\tilde{f}$ is perfect and irreducible.

A mapping $f : (X, \mathfrak{V}) \to (Y, \mathfrak{W})$ is called θ-uniformly regular if it is θ-uniformly continuous and, for every uniform θ-covering $v \in \mathfrak{V}$, there exists a uniform θ-covering $w \in \mathfrak{W}$ that is inscribed in the system $f^{\#} v$.

We shall call a θ-perfect irreducible mapping $f : (X, \mathfrak{V}) \to (Y, \mathfrak{W})$ a uniform θ-mapping if the mapping f is θ-uniformly continuous. Finally, a θ-uniformly regular uniform θ-mapping will be called a uniform regular θ-mapping.

Theorem 6. The uniform space $(X_{\mathfrak{V}}, \widetilde{\mathfrak{V}})$ is a projective object in the class of all θ-uniform spaces $(Y, \mathfrak{W})$ mapped onto the space $(X, \mathfrak{V})$ by uniform regular θ-mappings. Moreover, if $f : (Y, \mathfrak{W}) \to (X, \mathfrak{V})$ is a uniform regular θ-mapping, then in the commutative diagram

$$\begin{array}{ccc} Y_{\mathfrak{W}} & \xrightarrow{\tilde{f}} & X_{\mathfrak{V}} \\ \downarrow \pi_{\mathfrak{W}} & & \downarrow \pi_{\mathfrak{V}} \\ Y & \xrightarrow{f} & X \end{array}$$

the mapping $\tilde{f}$ is a uniformly continuous homeomorphism in both directions.

From this theorem follows the theorem on the θ-absolute (see ([1])), since every regular θ-mapping is a uniform regular θ-mapping with respect to the precompact θ-uniformities.

Faculty of Mechanics and Mathematics
M. V. Lomonosov Moscow State University

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* Here the closedness of the mapping f does not presuppose its continuity.

Note: Figure translations are in progress. See original paper for figures.