SOME APPROXIMATE METHODS FOR SOLVING TIME-OPTIMAL PROBLEMS IN BANACH SPACES IN THE PRESENCE OF PHASE CONSTRAINTS

1970

SovietRxiv

Abstract

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UDC 519.3

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SOME APPROXIMATE METHODS FOR SOLVING TIME-OPTIMAL PROBLEMS IN BANACH SPACES IN THE PRESENCE OF PHASE CONSTRAINTS

(Presented by Academician A. N. Tikhonov on 15 IV 1970)

1. Statement of the problem.

Let U be a given subset of a topological linear space Z; the elements $u \in U$ will be called controls. Let t be time, and t_0 a known initial instant of time. Suppose that for each $t \geq t_0$ there is given a set $\Omega[t_0, t]$ of functions $x(\tau)$, defined for $t_0 \leq \tau \leq t$ and taking their values in a given Banach space B, such that $\Omega[t_0, t] \subset \Omega[t_0, t']$ for all $t_0 \leq t' < t$. A function $x(\tau) \in \Omega[t_0, t]$ will be called a possible trajectory. Suppose that for all $t \geq t_0$ there is given a mapping that assigns to each $u \in U$ a possible trajectory $x(\tau, u) \in \Omega[t_0, t]$. Let $G(t)$, $t \geq t_0$, be given subsets of B, and let $I_\alpha(x(\tau), u, t)$ ($\alpha \in A$, a known set of indices) be functionals defined on the topological product $\Omega[t_0, t] \times U \times [t \geq t_0]$. We shall say that the trajectory $x(\tau, u) \in \Omega[t_0, t]$ satisfies the phase constraints if $x(\tau, u) \in G(\tau)$ for $t_0 \leq \tau \leq t$, $I_\alpha(x(\tau, u), u, t) \leq 0$ ($\alpha \in A$), and such a trajectory will be called admissible on $t_0 \leq \tau \leq t$. The subset of those $u \in U$ for which the corresponding trajectory $x(\tau, u)$ is admissible on $t_0 \leq \tau \leq t$ will be denoted by $U(t)$; the set of those $x \in B$ for which there exists $u \in U(t)$ such that $x(t, u) = x$ will be denoted by $X(t)$. Finally, let $Y(t)$, $t \geq t_0$, be given subsets of B. The time-optimal problem consists in finding such T^* and $u^* \in U(T^*)$ that $x(T^*, u^*) \in Y(T^*)$, while for all other T and $u \in U(T)$ for which $x(T, u) \in Y(T)$, the inequality $T \geq T^*$ holds. Such an instant T^* and control $u^* \in U(T^*)$ will be called optimal.

Everywhere below we shall assume that the following conditions, which we shall call conditions A, are satisfied: 1) the set U is convex and bicompact in the topology of the space Z; 2)

$$x(\tau, \alpha u + (1-\alpha)v) \equiv \alpha x(\tau, u) + (1-\alpha)x(\tau, v),$$

$t_0 \leq \tau \leq t$, for all $u, v \in U$, $0 \leq \alpha \leq 1$, and $t \geq t_0$; 3) for any net $\{u_k\} \subset U$

converging to u in Z ((^{1}), p. 29), the net $\{x(\tau, u_k)\} \subset B$ converges to $x(\tau, u)$ in the weak topology of B for all $\tau \geq t_0$; 4) the sets $G(t)$, $t \geq t_0$, are convex and closed in B; 5) the functionals $I_\alpha(x(\tau), u, t)$ are convex jointly in $(x(\tau), u) \in \Omega[t_0, t] \times U$, and $I_\alpha(x(\tau, u), u, t)$ are lower semicontinuous in $u \in U$ ((^{1}), p. 27); 6) the sets $Y(t)$, $t \geq t_0$, are convex and weakly bicompact in B for all $t \geq t_0$.

In describing and investigating the convergence of the methods given below for solving the time-optimal problem we shall need more stringent conditions, which we shall call conditions B: 1) conditions A are satisfied; 2)

$$\sup_{u \in U} \|x(t + \Delta t, u) - x(t, u)\|_B \to 0$$

as $\Delta t \to 0$; 3) for any sequence t_k, $t_k \leq t$, $t_k \to t$, and any $x_k \in G(t_k)$ such that $\|x - x_k\|_B \to 0$ $(k \to \infty)$, one has $x \in G(t)$; 4) if $I_\alpha(x(\tau, u), u, t) \leq 0$, then $I_\alpha(x(\tau, u), u, t') \leq 0$ for all $t_0 \leq t' < t$, and also

$$\lim_{\Delta \to +0} I_\alpha(x(\tau, u), u, t - \Delta t) \geq I_\alpha(x(\tau, u), u, t)$$

$(\alpha \in A)$; 5)

$$\sup_{y \in Y(t)} \min_{z \in Y(\tau)} \|y - z\| \to 0$$

as $\tau \to t - 0$, and

$$\sup_{\substack{z \in Y(\tau) \\ y \in Y(t)}} \min \|y - z\| \to 0$$

as $\tau \to t$.

2. Controllability and optimality criteria.

For

in the study of the time-optimal problem, an important role is played by the functional

$$M(c, t) = \min_{x \in X(t)} \min_{y \in Y(t)} (c, x - y) = \min_{u \in U(t)} (c, x(t, u)) - \max_{y \in Y(t)} (c, y),$$

where $c \in B^*$ is the space dual to B, and (c, z) is the value of the linear functional c on the element $z \in B$. Under conditions A, the set $X(t)$ is convex and weakly bicompact in B, and therefore $M(c, t)$ is defined for all $c \in B^*$, $t \geq t_0$.

Definition. We shall call the system $\{x(t,u)\}$ $U(T)$-controllable if there exists a $u \in U(T)$ such that $x(T,u) \in Y(T)$; otherwise the system is $U(T)$-uncontrollable.

Theorem 1. If conditions A are satisfied, then: 1) the system $\{x(t,u)\}$ is $U(T)$-controllable if and only if

$$M(c,T) = \min_{x \in X(T)} (c,x) - \max_{y \in Y(T)} (c,y)$$

for all $c \in B^*$; 2) T^* is optimal if and only if $M(c,t) \leq 0$ for all $c \in B^*$, and for any $t_0 \leq t < T^*$ there exists a $c_t \in B^*$ such that $M(c_t,t) > 0$.

Lemma 1. Let conditions A be satisfied. Then, for any $t \geq t_0$, the functional $M(c,t)$: 1) is concave in c; 2) $M(\alpha c,t) = \alpha M(c,t)$, $\alpha = \text{const} \geq 0$; 3) satisfies a Lipschitz condition in c in the norm of B^*; 4) is upper semicontinuous in c in the B-topology of the space B^*.

Lemma 2. Let conditions B be satisfied. Then: 1) $M(c,t)$, for any fixed $c \in B^*$, is lower semicontinuous in t and continuous from the left in t; 2) if, moreover, the system $\{x(t,u)\}$ is $U(T)$-controllable and $M(c,s) > 0$ for some $c \in B^*$, $t_0 \leq s < T$, then there exists a time t, $s < t \leq T$, such that $M(c,t) = 0$, $M(c,\tau) > 0$ for $s \leq \tau < t$.

3. Method I. To describe this method, introduce the functional

$$\rho(t) = \max_{\|c\| \leq 1} M(c,t).$$

From Lemma 1 and the bicompactness of the unit ball in B^* in the B-topology of B^* (Alaoglu' s theorem (^{2}), p. 459), it follows that $\rho(t)$ is defined for all $t \geq t_0$.

Theorem 2. If conditions A are satisfied, then: 1) the system $\{x(t,u)\}$ is $U(T)$-controllable if and only if $\rho(T) = 0$; 2) T^* is optimal if and only if $\rho(T^*) = 0$, $\rho(t) > 0$ for $t_0 \leq t < T^*$.

Lemma 3. If conditions B are satisfied, then $\rho(t)$ is lower semicontinuous for $t \geq t_0$ and continuous from the left for $t > t_0$.

We shall describe Method I under the assumption that condition B is satisfied and the system $\{x(t,u)\}$ is $U(T)$-controllable. Specify some sequence $\{\delta_k\}_{k=1}^{\infty}$, $\delta_k \geq 0$, $\delta_k \to 0$ $(k \to \infty)$, a constant $R > 0$, and a continuous function $\alpha(\rho)$, $\alpha(0) = 0$, $0 < \alpha(\rho) \leq R\rho$ for $\rho > 0$. As the initial approximation, take t_0 and such a $c_0 \in B^*$, $\|c_0\| \leq R$, that $\alpha(\rho(t_0)) \leq M(c_0,t_0)$ (naturally, one should assume $\rho(t_0) > 0$). Suppose the $(k-1)$-st approximation t_{k-1}, c_{k-1} is known and $t_0 < t_1 < \cdots < t_{k-1} < T$, $\rho(t) > 0$ for $t_0 \leq t \leq t_{k-1}$, $0 < \alpha(\rho(t_{k-1})) \leq M(c_{k-1},t_{k-1})$, $\|c_{k-1}\| \leq R$. Then determine t_k from the conditions $t_{k-1} < t_k \leq T$, $M(c_{k-1},t_k) \leq \delta_k$, $M(c_{k-1},\tau) > 0$ for $t_{k-1} \leq \tau < t_k$ (the existence of such a t follows from Lemma 3). By the choice of t_k, $\rho(t) > 0$ for

$t_{k-1} \le t < t_k$. Next, find c_k from the conditions $\|c_k\| \le R$, $\alpha(\rho(t_k)) \le M(c_k, t_k)$. If $M(c_k, t_k) = 0$, then $\alpha(\rho(t_k)) = 0$, $\rho(t_k) = 0$; the time t_k is optimal and the process is finished. If $M(c_k, t_k) > 0$, then $\rho(t_k) > 0$, and the process is continued. If this process does not terminate after a finite number of steps, then from Lemmas 1–3 and Theorem 2 it follows that

Theorem 3. Let condition B be satisfied and the system $\{x(t,u)\}$ be $U(T)$-controllable. Then the sequence t_k converges to the optimal T^*.

4. Method II. To describe Method II, introduce the functional

$$\chi(t) = \sup_{c \in \Pi(t)} M(c,t),$$

where

$$\Pi(t) = \{c : c \in B^*, \ \max_{y \in Y(t)} (c,y) = -1\}.$$

If conditions A are satisfied and $0 \notin Y(t)$, $0 \in X(t)$, then $\Pi(t)$ is nonempty and $\chi(t) > -\infty$; if $0 \in Y(t)$, then $\Pi(t)$ is empty and, by definition, we put $\chi(t) = -\infty$.

Theorem 4. Suppose conditions A are satisfied, and suppose $0 \in X(t)$ for all $t \ge t_0$. Then: 1) the system $\{x(t,u)\}$ is $U(T)$-controllable if and only if

when $\chi(T) \le 0$; 2) T^* is optimal if and only if $\chi(T^*) \le 0$, $\chi(t) > 0$ for $t_0 \le t < T^*$.

Lemma 4. If condition B is satisfied and $0 \in X(\tau)$ for all τ from a sufficiently small neighborhood of t, then $\chi(t)$ is lower semicontinuous at the point t. If, moreover, $0 \notin Y(t)$ and, for some $\omega_t > 0$, one of the sets $X(t) \cap [\omega_{tY}(t)]^0$ or $[X(t)]^0 \cap [\omega_{tY}(t)]^*$ is nonempty, then $\chi(t)$ is left-continuous at the point $t > t_0$.

We describe method II under the assumption that condition B is satisfied, the system $\{x(t,u)\}$ is $U(T)$-controllable, and $0 \in X(t)$ for all $t_0 \le t \le T$. Let a sequence $\delta_k \ge 0$ $(k = 1, 2, ...)$, $\delta_k \to 0$ $(k \to \infty)$, and a function $\alpha(\chi)$, $\alpha(\chi) < \chi$ for all χ, $\alpha(\chi) > 0$ for $\chi > 0$, be given. As the initial approximation take t_0 and $c_0 \in \Pi(t_0)$, $M(c_0, t_0) \ge \alpha(\chi(t_0)) > 0$. Suppose the $(k-1)$-st approximation t_{k-1}, $c_{k-1} \in \Pi(t_{k-1})$, $t_0 < t_1 < \cdots < t_{k-1} < T$, $\chi(t) > 0$ for $t_0 \le t \le t_{k-1}$, $0 < \alpha(\chi(t_{k-1})) \le M(c_{k-1}, t_{k-1})$ $(k \ge 1)$, is known. Then, by Lemma 3, there is a t_k, $t_{k-1} < t_k \le T$, such that $M(c_{k-1}, t_k) \le \delta_k$, $M(c_{k-1}, t) > 0$ for $t_{k-1} \le t < t_k$. Then $\chi(t) > 0$ for $t_{k-1} \le t < t_k$. It is possible that $0 \in Y(t_k)$; then t_k is the optimal time and the process is finished. If $0 \notin Y(t_k)$, then we find $c_k \in \Pi(t_k)$ from the condition $\alpha(\chi(t_k)) \le M(c_k, t_k)$. If $M(c_k, t_k) \le 0$, then $\chi(t_k) \le 0$ and t_k is optimal; the process is finished. If $M(c_k, t_k) > 0$, then $\chi(t_k) > 0$ and we continue the process. If this process does not end in a finite number of steps, then Lemma 4 and Theorem 4 imply

Theorem 5. Suppose condition B is satisfied, the system $\{x(t,u)\}$ is $U(T)$-controllable, $0 \in X(t)$ for $t_0 \leq t \leq T$, and suppose that for some $\omega_t > 0$ one of the sets $X(t) \cap [\omega_{tY}(t)]^0$ or $[X(t)]^0 \cap [\omega_{tY}(t)]^*$ is nonempty for $t_0 < t \leq T$. Then the sequence t_k $(k = 1, 2, \ldots)$, obtained by method II, converges to the optimal T^*.

In practice, apparently, instead of $\chi(t)$ it is more convenient to work with the functional

$$\omega(t) = \inf_{c \in \Pi(t)} \max_{x \in X(t)} (-c, x).$$

Since $\chi(t) = 1 - \omega(t)$, it is not difficult to set out method II and formulate Theorems 4, 5 and Lemma 4 using $\omega(t)$.

5. Applications. Example 1. Let the process be described by the system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \geq 0, \qquad x(0) \in X_0,$$

where $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^r)$, A and B are constant matrices of dimensions $n \times n$ and $n \times r$, respectively; X_0 is a given subset of the Euclidean space E_n. Take U: either

$$U_1 = \{u(t) : \operatorname{vrai\,sup}_{0 \leq t \leq T} |u^i(t)| \leq \alpha_i = \text{const}, \quad i = 1, \ldots, r\},$$

or

$$U_2 = \left\{ u(t) : \int_0^T u^2(t)\, dt \leq \alpha = \text{const} \right\},$$

or $U_3 = U_1 \cap U_2$. It is required to transfer the system in minimum time from the set X_0 to the set $Y(t)$, observing the phase constraints:

$$x(t) \in G(t), \quad t \geq 0, \qquad \int_0^T x^2(t)\, dt \leq \beta = \text{const};$$

here $Y(t)$, $G(t)$ are given subsets of E_n.

Theorem 6. Suppose 1) the system is $U(T)$-controllable for some $T < \infty$; 2) X_0 is convex, closed, bounded; 3) $G(t)$ is convex, closed and $G(t-0) \subset G(t)$ for all $t \geq 0$; 4) $Y(t)$ is convex, closed, bounded, left-continuous in the Hausdorff sense, and $Y(t+0) \subset Y(t)$, $t \geq 0$. Then the sequence $\{t_k\}$ from method I converges to the optimal T^*. If, in addition to 1)–4), the following conditions are satisfied: 5) the rank of the matrix $\{B, AB, \ldots, A^{n-1}B\}$ is equal to n; 6) for any $T > 0$ there exists a ball $K_T \subset E_n$ with center at zero such that $K_T \subset G(t)$

for $t_0 \le t \le T$, then the sequence $\{t_k\}$ from method II converges to the optimal T^*.

* Here $[\,]^0$ denotes the interior of a set.

Some variants of methods I, II have been studied, for example, in ($^{3-6}$).

Example 2. Let the process be described by the conditions:

$$x_t = x_{ss}, \quad x \equiv x(s,t), \quad (s,t) \in Q_T = \{0 \le s \le 1,\ 0 \le t \le T\},$$

$$x_s(0,t) = 0, \quad x_s(1,t) = \alpha[u(t) - x(1,t)], \quad x(s,0) = 0, \quad \alpha = \text{const} > 0.$$

From these conditions, for each fixed $u = u(t) \in U(T) = \{u(t) : u(t) \in L_2[0,T],\ \operatorname{vrai\,sup}_{0 \le t < T} |u(t)| \le 1\}$, $x = x(s,t,u)$ is uniquely determined (^{7}). It is required, in the minimal time T, to achieve the fulfillment of

$$x(s,t,u) \in Y = \left\{ y(s) : y(s) \in L_2[0,1], \int_0^1 |y(s) - y_0(s)|^2\, ds \le \delta^2 \right\}, \quad \delta = \text{const} > 0,$$

where $y_0(s)$ is a given function from $L_2[0,1]$. Here one must take

$$M(c,t) = \min_{u \in U(T)} \min_{y \in Y} \int_0^1 c(s)[x(s,t,u) - y(s)]\, ds, \quad c(s) \in L_2[0,1].$$

Theorem 7. *If, for some $T < \infty$, the system is $U(T)$-controllable, then the sequence $\{t_k\}$, determined by method I or II, converges to the optimal T (method II cf. (^{8}), pp. 303, 380; for method I see (^{9})).*

Example 3. Let the process be described by the conditions:

$$x_{tt} = a^2 x_{ss} + u_0(s,t), \quad x \equiv x(s,t), \quad (s,t) \in Q_T \quad (a = \text{const} > 0),$$

$$x(0,t) = x(1,t) = 0, \quad x(s,0) = u_1(s), \quad x_t(s,0) = u_2(s).$$

From these conditions, for each $u = (u_0, u_1, u_2) \in U = U_0 \times U_1 \times U_2$,

$$U_0 = \{u_0(s,t) : \|u_0\|_{L_2(Q_T)} \le \alpha_0\}, \quad U_1 = \{u_1(s) : \|u_1\|_{W_2^{(2)}[0,1]} \le \alpha_1,\ u_1(0) = u_1(1) = 0\},$$

$$U_2 = \{u_2(s) : \|u_2\|_{W_2^{(1)}[0,1]} \le \alpha_2,\ u_2(0) = u_2(1) = 0\}, \quad \alpha_i = \mathrm{const} \ge 0,\ i = 0,1,2,$$

the solution $x = x(s,t,u) \in W_2^{(2)}(Q_T)$ is uniquely determined $(^{10})$. It is required, in the minimal time T, to achieve the fulfillment of

$$\int_0^1 \left(|x(s,T,u) - \bar{y}_0(s)|^2 + |x_t(s,T,u) - \bar{y}_1(s)|^2\right) ds \le \delta,$$

while observing the phase constraints

$$\beta_1 \int_0^1 |x(s,T,u) - \bar{y}_2(s)|^2 \, ds + \beta_2 \int_0^1 |x_t(s,t,u) - \bar{y}_3(s)|^2 \, ds \le \beta_3,$$

where $\bar{y}_i(s) \in L_2[0,1]$ are given functions, $\delta = \mathrm{const} \ge 0$, $\beta_i = \mathrm{const} \ge 0$. Let us take

$$c = (c_1(s), c_2(s)), \quad \|c(s)\| = \|c\|_{L_2^{(2)}[0,1]} = \left(\int_0^1 (|c_1(s)|^2 + |c_2(s)|^2)\, ds\right)^{1/2}, \quad y = (y_1(s), y_2(s));$$

$$\bar{y} = (\bar{y}_0(s), \bar{y}_1(s)), \quad Y = \{y : \ \|y - \bar{y}\|_{L_2^{(2)}[0,1]} \le \delta\},$$

$$M(c,t) = \min_{u \in U(t)} \min_{y \in Y} \int_0^1 (c_1(s)x(s,t,u) + c_2(s)x_t(s,t,u) - c_1(s)\bar{y}_1(s) -$$

$$-c_2(s)\bar{y}_2(s))\, ds.$$

Theorem 8. *If, for some $T < \infty$, the system is $U(T)$-controllable, then the sequence $\{t_k\}$ from method I converges to the optimal T^*.*

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Received
13 IV 1970

REFERENCES

[1] R. Edwards, *Functional Analysis*, Moscow, 1969.
[2] N. Dunford, J. T. Schwartz, *Linear Operators. General Theory*, IL, 1962.
[3] A. B. Rabinovich, Zh. Vychisl. Mat. i Mat. Fiz., 6, No. 3, 433 (1966).
[3] T. Fujisawa, Y. Yasuda, SIAM J. Control, 5, No. 4, 501 (1967).
[5] T. G. Babunashvili, DAN, 155, No. 2, 295 (1964).
[6] F. P. Vasil' ev, R. P. Ivanov, Zh. Vychisl. Mat. i Mat. Fiz., 10, No. 5, 1124 (1970).
[7] Yu. V. Egorov, Zh. Vychisl. Mat. i Mat. Fiz., 3, No. 5, 887 (1963).
[8] A. G. Butkovskii, *Theory of Optimal Control of Systems with Distributed Parameters*, "Nauka," 1965.
[9] F. P. Vasil' ev, Zh. Vychisl. Mat. i Mat. Fiz., 10, No. 3 (1970).
[10] O. A. Ladyzhenskaya, *Mixed Problem for a Hyperbolic Equation*, Moscow, 1953.

Note: Figure translations are in progress. See original paper for figures.

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