ON THE EXISTENCE OF WEIGHTED FINITE INCIDENCE STRUCTURES

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Abstract

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A. Ya. Petrenyuk

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The article introduces the concept of a weight function of a finite incidence structure; with the aid of this concept, necessary conditions are found for the existence of l-weighted structures, defined below.

Let v be a natural number, $v \geq 4$, and let E be a set consisting of v elements. The symbol $P(E)$ denotes the collection of subsets of the set E. Let $n(\beta)$ be a function defined on $P(E)$ and taking values in the set Z^+ of nonnegative integers. This function defines on E a (finite) incidence structure (briefly: a structure), containing the subset $\beta \subseteq E$ exactly $n(\beta)$ times. The elements of the set $P(E)$ that enter the structure are called blocks, and the function $n(\beta)$ is called the function of block multiplicities in the structure. If H is a finite set, then $|H|$ often denotes the number of elements in H; the number $|\beta|$ is called the volume of the block β.

Each structure defines on $P(E)$ a function

$$\rho(X) = \sum_{\{\beta : X \subseteq \beta\}} n(\beta), \tag{1}$$

which we shall call the weight function of the structure.

Let $K = \{k_1, \dots, k_s\}$ be a set of natural numbers; let l be a natural number, $2 \leq l < k_1 < \dots < k_s < v$; let $P_l(E) = \{L : L \subset E,\ |L| = l\}$, and suppose that on $P_l(E)$ a function $\lambda(L)$ with values in Z^+ is given. A structure for each block β of which the inclusion $|\beta| \in K$ holds and in which for every $L \in P_l(E)$ the equality

$$\rho(L) = \lambda(L), \tag{2}$$

holds is called an l-weighted structure of type $C(K, l, \lambda(L), E)$. The substructure A_i of an l-weighted structure A, defined by the block-multiplicity function

$$n_i(\beta) = \begin{cases} n(\beta), & \text{if } |\beta| = k_i, \\ 0 & \text{otherwise,} \end{cases}$$

is called the i-th equiblock component of the structure A; the weight function of A_i is denoted by $\rho_i(X)$.

Counting in two different ways the number of occurrences of subsets D such that $T \subseteq D \subset E$, in the i-th equiblock component, we obtain that the equality

$$\binom{k_i - t}{d - t} \rho_i(T) = \sum_{\substack{H \subseteq E \setminus T \\ |H| = d - t}} \rho_i(T \cup H), \tag{3}$$

is valid, where d is a natural number, $0 \le |T| = t < d \le k_i,\ T \subset E$.

Putting $d = l$ in (3) and summing the equalities (3) with respect to the index i, taking into account (2) and the obvious formula

$$\rho(X) = \sum_{i=1}^{s} \rho_i(X),$$

we obtain the following theorem.

Theorem 1. *Suppose there is an l-weighted structure of type $C(K, l, \lambda(L), E)$. Then the relation*

$$\sum_{i=1}^{s} \binom{k_i - t}{l - t} \rho_i(T) = \sum_{\substack{H \subseteq E \setminus T \\ |H| = l - t}} \lambda(T \cup H) \tag{4}$$

holds for every $T,\ T \subset E,\ 0 \le |T| = t < l$.

We shall denote by

$$d \left[\binom{k_i - t}{l - t} \right]_{i=1}^{s}$$

the greatest common divisor of the integers

$$\binom{k_1 - t}{l - t}, \dots, \binom{k_s - t}{l - t}.$$

Theorem 2. *For the existence of an l-weighted structure of type $C(K, l, \lambda(L), E)$, it is necessary that the conditions*

$$\sum \lambda(T \cup H) \Big/ d \left[\binom{k_i - t}{l - t} \right]_{i=1}^{s} = \text{an integer} \tag{5}$$

hold for all $T,\ T \subset E,\ 0 \le |T| = t < l$. *The summation in (5) is taken over all* $H,\ H \subset E \setminus T,\ |H| = l - t$.

Condition (5) follows directly from (4).

In the case when the function $\lambda(L)$ is identically equal to the natural number λ, an l-weighted structure of type $C(K, l, \lambda, E)$ is called ***l*-balanced**. For $s = 1$ the latter is a tactical configuration ([1,3]), which for $l = 2$ is called a balanced incomplete block design ([2]). Condition (5) for the case of l-balanced structures takes the form

$$\lambda \binom{v-t}{l-t} \Big/ d\left[\binom{k_i - t}{l-t}\right]_{i=1}^{s} = \text{an integer}, \quad t = 0, 1, \dots, l-1. \tag{6}$$

Relation (4) for $\lambda(L) \equiv \lambda,\ l = 2$ and $t = 0$ occurs in ([2]). Conditions (6) for $s = 1$ turn into the known existence conditions for tactical configurations ([1]).

Faculty of Mechanics and Mathematics
M. V. Lomonosov Moscow State University

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Note: Figure translations are in progress. See original paper for figures.