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Abstract

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ON THE TRIANGULAR REPRESENTATION OF OPERATORS CLOSE TO UNITARY ONES, AND THE MULTIPLICATIVE FACTORIZATION OF THEIR CHARACTERISTIC FUNCTIONS

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Let $\mathfrak{H}$ be a separable Hilbert space, $\mathfrak{R}$ the ring of bounded linear operators acting in $\mathfrak{H}$, and $\mathfrak{S}_\omega$ a symmetrically normed (s.n.) ideal of the ring $\mathfrak{R}$, introduced by V. I. Matsaev (^{1}).

I. Ts. Gohberg and M. G. Krein in the paper (^{2}) obtained a triangular representation of operators $T \in \mathfrak{R}$ having unitary spectrum. In the same paper, as well as in the work (^{3}), multiplicative factorizations of the characteristic functions of these operators are given. The results listed are extended in the present paper to operators $T \in \mathfrak{R}$ which, generally speaking, do not have unitary spectrum.

Lemma 1. *If an invertible operator $T \in \mathfrak{R}$ possesses a discrete $*$ chain $\mathfrak{P} = \{P_j\}_0^\infty$, then there exists an operator $V \in \mathfrak{R}$ possessing the chain $\mathfrak{P}$ and satisfying the conditions $\Delta P_j V \Delta P_j = 0$ $(\Delta P_j = P_j - P_{j-1},\ j = 1, 2, ...)$, such that*

$$T = D(I + V)\left(D = \int_{(\mathfrak{P})} e^{i\varphi(P)}\, dP\right),$$

where the integral converges strongly, and the scalar function $\varphi(P)$ is determined from the relations $e^{i\varphi(P_j)}\Delta P_j = \Delta P_{jT}\Delta P_j$. If, in addition, $I - T^*T \in \mathfrak{S}_\omega$, then the operator V is Volterra.

Proof. The strong convergence of the integral D was proved in the paper (^{4}). Let g_j $(j = 1, 2, ...)$ be a unit vector of the one-dimensional subspace $\Delta P_j\mathfrak{H}$. Since

$$Tg_j = \beta_{j,1}g_1 + ... + \beta_{j,j-1}g_{j-1} + e^{i\varphi(P_j)}g_j,$$

the numbers $e^{i\varphi(P_j)}$ belong to the spectrum of the operator T. By the invertibility of T,

$$\inf_j |e^{i\varphi(P_j)}| > 0,$$

and, consequently, the operator D is invertible.

Put $V = D^{-1}T - I$. Then $T = D(I + V)$ and

$$VP_j = D^{-1}P_{jTP}j - P_j = P_j(D^{-1}T - I)P_j = P_{jVP}j,$$

$$\Delta P_{jV}\Delta P_j = \Delta P_{jD}^{-1}\Delta P_{jT}\Delta P_j - \Delta P_j = 0.$$

Let us note that

$$\Delta P_j(I + V^*)^{-1}\Delta P_j = \Delta P_j, \qquad \Delta P_{jD}^* D(I + V)\Delta P_j = \Delta P_{jD}^* D\Delta P_j.$$

From the equalities

$$I - D^*D = \sum_{j=1}^{\infty} \Delta P_j(I - D^*D)\Delta P_j =$$

$$= \sum_{j=1}^{\infty} \Delta P_j((I + V^*)^{-1} - D^*D(I + V))\Delta P_j =$$

$$= \sum_{j=1}^{\infty} \Delta P_j(I + V^*)^{-1}H\Delta P_j \qquad (H = I - T^*T)$$

it follows (see also (6), p. 109) that $I - D^*D \in \mathfrak{S}_\omega$. Hence one can obtain the relation

$$S = (I + V^*)^{-1}(I - (I + V^*)(I + V)) \in \mathfrak{S}_\omega.$$

Moreover, $\Delta P_j S \Delta P_j = 0$, and therefore, by (1), the integral

$$\int_{[\mathfrak{P}]} PS\,dP = \sum_{j=1}^{\infty} P_{j-1}(I + V^*)^{-1}\Delta P_j - \sum_{j=1}^{\infty} P_{j-1}(I + V)\Delta P_j =$$

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* For terminology and notation see (4,5).

$$= -\sum_{j=1}^{\infty} P_{j-1} V \Delta P_j = -V$$

converges uniformly and is a Volterra operator. The lemma is proved.

Let $T \in \mathfrak{R}$ and $I - T^*T \in \mathfrak{S}_\omega$. Denote by $\mathfrak{H}_0$ the closure of the linear span of all root subspaces of the operator T corresponding to its eigenvalues not lying on the unit circle. Introduce the orthoprojectors $P^{(0)}$ and $P^{(1)}$ respectively onto $\mathfrak{H}_0$ and $\mathfrak{H}_1 = \mathfrak{H} \ominus \mathfrak{H}_0$. Consider the operator T_0, induced by the operator $P^{(0)}TP^{(0)}$ in $\mathfrak{H}_0$, and the operator T_1, induced by the operator $P^{(1)}TP^{(1)}$ in $\mathfrak{H}_1$. The operator T_0 has a discrete chain $\mathfrak{P}^{(0)} = \{Q_j\}_0^\infty$*, while the operator T_1** has a maximal chain $\mathfrak{P}^{(1)} = \{Q\}$, separating its spectrum (2). We shall also need the maximal chain $\mathfrak{P} = \{P\}$, which is the union of the chains $\mathfrak{P}_0 = \{QP^{(0)}\}$ $(Q \in \mathfrak{P}^{(0)})$ and $\mathfrak{P}_1 = \{P^{(0)} + QP^{(1)}\}$ $(Q \in \mathfrak{P}^{(1)})$, and the scalar function $\varphi(P)$ $(P \in \mathfrak{P})$, defined by the following conditions:

I. $e^{i\varphi(P_j)}\Delta P_j = \Delta P_{jT}\Delta P_j$ $(\Delta P_j = P_j - P_{j-1},\ P_j \in \mathfrak{P}_0)$.

II. $\varphi(P)$ $(P \in \mathfrak{P}_1)$ coincides with the least of the numbers τ for which the arc e^{it} $(0 \le t \le \tau)$ contains the spectrum of the operator $T_{1P}h = PT_1h$ acting in the subspace $P\mathfrak{H}_1$ $(h \in P\mathfrak{H}_1)$.

Theorem 1. If the operator $T \in \mathfrak{R}$ is invertible and $I - T^*T \in \mathfrak{S}_\omega$, then

$$T = \int_{[\mathfrak{P}]} e^{i\varphi(P)}\, dP \left(I + \int_{[\mathfrak{P}]} (I - PHP)^{-1} PH\, dP \right)^{-1} \quad (H = I - T^*T), \quad (1)$$

where the first integral converges strongly, and the second uniformly.

Proof. Using Lemma 1 and Theorem 1 of paper (2), it is not difficult to represent the operator T in the form $T = D(I + V)$, where

$$D = \int_{\mathfrak{P}} e^{i\varphi(P)}\, dP,$$

and V is a Volterra operator possessing the chain $\mathfrak{P}$. From Theorems 3.1 and 7.1 of paper (5) follows the equality

$$(I + V)^{-1} = I + \int_{[\mathfrak{P}]} (I - PHP)^{-1} PH\, dP.$$

Theorem 1 shows that an invertible operator $T \in \mathfrak{R}$ satisfying the condition $I - T^*T \in \mathfrak{S}_\omega$ can be represented as the sum of a normal operator and a Volterra operator possessing one and the same maximal chain.

Let

$$T \in \mathfrak{R}, \qquad J_T = \operatorname{sign}(I - T^*T), \qquad \mathfrak{K}_T = \overline{(I - T^*T)\mathfrak{H}}.$$

The function

$$\theta_T(\zeta) = \left(T - \zeta J_{T^*} |I - TT^*|^{1/2} (I - \zeta T^*)^{-1} |I - T^*T|^{1/2}\right)\Big|_{\mathfrak{K}T},$$

acting from the space $\mathfrak{R}_T$ into the space $\mathfrak{R}_{T^*}$, is called the characteristic function of the operator T.

Lemma 2. If the operator $T \in \mathfrak{R}$ is invertible and $I - T^*T \in \mathfrak{S}_\omega$, then

$$\theta_T(\xi) = U_0 \left(I - J_T |H|^{1/2} F P^{(1)} (D^*D)^{-1/2} (\xi D^* + (D^*D)^{1/2})(I - \xi T^*)^{-1} |H|^{1/2}\right)$$
$$\times \left(I - J_T |H|^{1/2} F P^{(0)} (D^*D)^{-1/2} (\xi D^* + (D^*D)^{1/2})(I - \xi T^*)^{-1} |H|^{1/2}\right) \Big|_{\mathfrak{R}_T}$$
$$(F = (I + (D^*D)^{1/2}(I + V))^{-1}).$$

The operator U_0 satisfies the relation $U_0 J_{TU} 0^* = J_{T^*}$, maps the whole space $\mathfrak{R}_T$ onto the whole space $\mathfrak{R}_{T^*}$, and is computed by the formula

$$U_0 = (T^*)^{-1}(I - J_T |H|^{1/2} F^* |H|^{1/2}).$$

Let $\mathfrak{S}$ be an arbitrary s.n. ideal of the ring $\mathfrak{R}$.

Lemma 3. If the operator $T \in \mathfrak{R}$ is invertible and the operator $I - T^*T$ belongs simultaneously to the s.n. ideals $\mathfrak{S}$ and $\mathfrak{S}_\omega$, then

$$I - J_T |H|^{1/2} F P^{(0)} (D^*D)^{-1/2} (\xi D^* + (D^*D)^{1/2})(I - \xi T^*)^{-1} |H|^{1/2} =$$
$$= \prod_{j=1}^{\infty} \left(Q^{(j)} \frac{\xi_j - \xi}{1 - \xi \bar{\xi}_j} \frac{|\xi_j|}{\xi_j} P^{(j)} \right), \tag{2}$$

where the infinite product converges in the norm of $\mathfrak{S}$, the numbers ξ_j ($j = 1, 2, ...$) are determined from the relations $\xi_j \Delta P_j = \Delta P_{jT} \Delta P_j$, $P^{(j)}$ ($j = 1, 2, ...$) are one-dimensional projectors computed by the formula

$$P^{(j)} = \frac{1 + \xi_j}{1 - \bar{\xi}_j} J_T |H|^{1/2} F \Delta P^*_{jF} |H|^{1/2}$$

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* For definiteness, we consider only the case where $\dim \mathfrak{H}_0 = \infty$.

** The spectrum of the operator T_1 lies on the unit circle (6).

and $Q^{(j)} = I - P^{(j)}$.

Lemma 4. If the operator $T \in \mathfrak{R}$ is invertible and the operator $I - T^*T$ belongs simultaneously to the s.n. ideals $\mathfrak{S}$ and $\mathfrak{S}_\omega$, then

$$I - J_T |H|^{1/2} F P^{(1)} (D^*D)^{-1/2} (\xi D^* + (D^*D)^{1/2})(I - \xi T^*)^{-1} |H|^{1/2} = \\ = \int_{(\mathfrak{P}_1)} \left(I - \frac{e^{i\varphi(P)} + \xi}{e^{i\varphi(P)} - \xi} \, d\left(2 J_T |H|^{1/2} F P^{(1)} P P^{(1)} F^* |H|^{1/2}\right) \right). \qquad (3)$$

The integral (3) converges in the norm of $\mathfrak{S}$.

Lemma 4 for the special case when $\|H\| \le 1$ and the spectrum of the operator T lies on the unit circle was given in paper (5).

Let the function $\theta(\xi)$, whose values are linear bounded operators in $\mathfrak{H}$, satisfy the following conditions: 1) $\theta(\xi)$ is holomorphic in the domain obtained by deleting from the disk $|\xi| < 1$ at most a countable set of isolated points $\{\xi_j\}_{j=1}^{q}$ ($q \le \infty$, $\xi_j \ne 0$); 2) $\theta(\xi)$ is invertible at least at one point ξ ($|\xi| < 1$); 3) $\|\theta(\xi)\| \le 1$ ($|\xi| < 1$) and $\|\theta(0) f\| < \|f\|$ for all $f \ne 0$.

Then, as shown in article (7), there exists a contraction T such that $\theta(\xi) = U_* \theta_T(\xi) U$, where the operators U and U_* isometrically map the spaces $\mathfrak{H}$ and $\mathfrak{R}_{T^*}$, respectively, onto $\mathfrak{R}_T$ and $\mathfrak{H}$. If, in addition: 4) there exists a point ξ_0 ($|\xi_0| < 1$) such that the operator $I - \theta^*(\xi_0)\theta(\xi_0)$ belongs simultaneously to the ideals $\mathfrak{S}$ and $\mathfrak{S}_\omega$, then (see (2)) the operator $I - T^*T$ belongs to these ideals. Hence from Lemmas 2, 3, and 4 it follows that

Theorem 2. If the function $\theta(\xi)$, whose values are linear bounded operators in $\mathfrak{H}$, satisfies conditions 1)–4), then

$$\theta(\xi) = U_{(\omega)} \int_{(\mathfrak{P}_1)} \left(I - \frac{e^{i\varphi(P)} + \xi}{e^{i\varphi(P)} - \xi} \, dF(P) \right) \prod_{j=1}^{q} \left(Q^{(j)} + \frac{\xi_j - \xi}{1 - \xi \bar{\xi}_j} \frac{|\xi_j|}{\xi_j} P^{(j)} \right) U, \quad (4)$$

where $\mathfrak{P}_1$ is a maximal chain in some space $\mathfrak{H}_1 \subset \mathfrak{H}$; $\varphi(P)$ ($P \in \mathfrak{P}_1$, $0 \le \varphi(P) \le 2\pi$) is a nondecreasing scalar function; $F(P)$ ($P \in \mathfrak{P}_1$) is a positive operator-valued function; $P^{(j)}$ ($j = 1, 2, \ldots$) are one-dimensional orthoprojections; $Q^{(j)} = I - P^{(j)}$, and $U_{(0)}$ is an isometric operator mapping the space $\mathfrak{H}_T$ onto $\mathfrak{H}$. The integral and the infinite product converge in the norm of $\mathfrak{S}$.

Theorem 2 generalizes results of V. P. Potapov (^{8}) and Yu. P. Ginzburg (^{9}, ^{10})* on the multiplicative representation of contractive analytic operator-valued functions.

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* In the paper ([11]), Yu. P. Ginzburg obtained a multiplicative representation of functions not contained in the class of functions considered in Theorem 2.

Note: Figure translations are in progress. See original paper for figures.

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