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Abstract

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MATHEMATICS

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ON WEIGHTED APPROXIMATION BY POLYNOMIALS ON THE REAL AXIS

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Approximation by polynomials with the weight $\rho(x) = e^{-x^2/2}$ is considered. Denote by $\|f\|_p$ the norm in the space $\mathcal{L}_p = \mathcal{L}_p(-\infty, +\infty)$, and let $\mathcal{L}_p^*$ be the set of such functions $f(t)$ that $f\rho \in \mathcal{L}_p$ and $\|f\|_p^* = \|\rho f\|_p$. Let n be an arbitrary integer ≥ 4, $\nu = [n/2]$, π_n the set of algebraic polynomials of degree not exceeding n, and $c_1, c_2, ...$ absolute constants.

We consider the behavior of the quantities

$$\varepsilon_n^{(p)*}(f) = \inf_{\varphi_n \in \pi_n} \|f - \varphi_n\|_p^*. \tag{1}$$

Theorem 1. If $F(t) = \int f(\tau)d\tau$, and $f \in \mathcal{L}_p^*$, then

$$\varepsilon_n^{(p)*}(F) \leq c_1 n^{-1/2} \varepsilon_{n-1}^{(p)*}(f). \tag{2}$$

Corollary. The relation

$$\varepsilon_n^{(p)*}(F) \leq c_2 n^{-1/2} \|f\|_p^* \tag{3}$$

holds.

Already the corollary (3) refines some known results. For $n = +\infty$ and under the additional condition $\|F\|_\infty + \|f\|_\infty < \infty$, (3) follows from a theorem of M. M. Dzhrbashyan ([2]), and under the somewhat weaker condition $\|f\|_\infty < \infty$–from a result of the author ([5]). A. S. Dzhafarov ([1]) considers the case $p < +\infty$. We note that in the works ([1],[2],[5]) more general weight functions are considered.

We shall prove Theorem 1 at the end of the article, after first establishing the corollary (3).

Theorem 2. Suppose $F(t)$ has bounded variation on every finite interval; then (the right-hand side is assumed finite)

$$\varepsilon_n^{(1)*}(F) \leq c_3 n^{-1/2} \int_{-\infty}^{+\infty} \rho(t)\,|dF(t)|. \tag{4}$$

Proof of Theorem 2. By the duality theorem of S. M. Nikolsky ([3]), we have

$$\varepsilon_n^{(1)*}(F) = \sup_{g \in B_n} \int F(t) g(t) \rho(t)\,dt = \sup_{\rho^{-1} G \in B_n} \int G(t)\,dF(t), \tag{5}$$

where B_n is the set of such functions that $g(t) \in \mathcal{L}_\infty$, $\|g\|_\infty \leq 1$, and $\int g\varphi_n \rho\,dt = 0$ for any $\varphi_n \in \pi_n$, and

$$G(x) = \int_x^\infty g(t)\rho(t)\,dt = \int_{-\infty}^{+\infty} \Gamma_x(t) g(t) \rho(t)\,dt = \int (\Gamma_x - \varphi_n) g\rho\,dt \quad (\varphi_n \in \pi_n), \tag{6}$$

where $\Gamma_x(t) = 0$ for $t < x$, and $\Gamma_x(t) = 1$ if $t \geq x$.

It follows from ([6]) that for any x there exists $\varphi_{nx} \in \pi_n$ such that

$$^*\|\Gamma_x - \varphi_{nx}\|_1^* \leq c_3 n^{-1/2} \rho(x).$$

From relation (6), taking $\varphi_n = \varphi_{nx}$, we obtain that $|G(x)| \leq c_3 n^{-1/2} \rho(x)^*$ for every admissible G. Thus relation (4) follows from (5), as was required to prove.

Let $F_n(f;t)$ be the $(C,1)$-means of order n of the expansion of a certain funct–

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* In the case $|x| \leq \sqrt{n}/4$, the lemma of § 2 of the work ([6]) is used; if $x > \sqrt{n}/4$, then $\varphi_{nx}(t) \equiv 0$, and if $x < -\sqrt{n}/4$, then $\varphi_{nx}(t) \equiv 1$.

the expansion of $f \subset \mathcal{L}_1^*$ in orthogonal Hermite polynomials, and

$$v_n(f;t) = (n - \nu + 1)^{-1} \left[(n+1) f_n(f;t) - \nu F_\nu(f;t) \right] \tag{7}$$

are the Vallée-Poussin means. Then $v_n(\varphi_\nu; t) \equiv \varphi_\nu(t)$ for every $\varphi_\nu \in \pi_\nu$. From § 5 of paper ([4]) it follows that $\|F_n(f;t)\|_\infty^* \leq c_4 \|f(t)\|_\infty^*$; thus from (7) we obtain

$$\|v_n(f;t)\|_\infty^* \leq c_5 \|f\|_\infty^* \tag{8}$$

and further

$$\|v_n(f;t)\|_1^* = \sup_{\|g\|_\infty^* \le 1} \int g(t) v_n(f;t) \rho^2(t)\, dt =$$

$$= \sup_{\|g\|_\infty^* \le 1} \int f(t) v_n(g;t) \rho^2(t)\, dt \le c_5 \|f\|_1^*. \tag{9}$$

Proof of inequality (3). Let first $p = \infty$. Put $\psi_x(t) = e^{t^2}$ for $t \in [0, x]$ and $\psi_x(t) = 0$ for $t \notin [0, x]$. By Theorem 2 there exists a polynomial $\varphi_{\nu x} \in \pi_{\nu-1}$ such that

$$\|\psi_x - \varphi_{\nu x}\|_1^* \le c_6 n^{-1/2} \rho^{-1}(x).$$

If $\varphi_\nu \in \pi_{\nu-1}$, then $\int (f - v_{n-1}) \varphi_\nu \rho^2 \, dt = 0$, and

$$\left| \int_0^x [f(t) - v_{n-1}(f;t)]\, dt \right| = \left| \int_{-\infty}^{+\infty} [f(t) - v_{n-1}(f;t)][\psi_x(t) - \varphi_{\nu x}(t)] \rho^2(t)\, dt \right| \le$$

$$\le \|f - v_{n-1}(f)\|_\infty^* \, \|\psi_x - \varphi_{\nu x}\|_1^* \le (c_5 + 1) \|f\|_\infty^* \, \|\psi_x - \varphi_{\nu x}\|_1^* \le c_6 n^{-1/2} \|f\|_\infty^* \rho^{-1}(x).$$

Thus,

$$\varepsilon_n^{(\infty)*}(F) \le c_6 n^{-1/2} \|f\|_\infty^*,$$

and since $v_n(\varphi_\nu; t) \equiv \varphi_\nu(t)$ for any $\varphi_\nu \in \pi_\nu$, from relation (8) we obtain

$$\|F(t) - v_n(F;t)\|_\infty^* \le (c_5 + 1) \varepsilon_\nu^{(\infty)*}(F) \le c_7 n^{-1/2} \|f\|_\infty^*, \tag{10}$$

whence inequality (3) follows for $p = +\infty$.

From inequality (9) and Theorem 2 we obtain

$$\|F(t) - v_n(F;t)\|_1^* = \inf \|F(t) - \varphi_\nu(t) + v_n(F - \varphi_\nu; t)\|_1^* \le$$

$$\le (1 + c_4) \inf \|F(t) - \varphi_\nu(t)\|_1^* \le c_8 n^{-1/2} \|f\|_1^*. \tag{11}$$

By the Riesz-Thorin interpolation theorem (see ([7]), vol. II), from relations (10) and (11) we obtain

$$\|F(t) - v_n(F;t)\|_p^* \le c_2 n^{-1/2} \|f\|_p^*. \tag{12}$$

Inequality (3) is proved.

Proof of Theorem 1. Let $\varphi_{n-1} \in \pi_{n-1}$ and

$$\|f - \varphi_{n-1}\|_p^* < 2\varepsilon_n^{(p)}(f).$$

If in inequality (3) we replace the function f by $f - \varphi_{n-1}$, then for a suitably chosen polynomial we have

$$\left\| F(t) - \int^t \varphi_{n-1}(\tau)\, d\tau - \psi_n(t) \right\|_p^* \le c_2 n^{-1/2} \|f - \varphi_{n-1}\|_p^* \le 2c_2 n^{-1/2} \varepsilon_{n-1}^{(p)*}(f),$$

which completes the proof of the theorem.

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Note: Figure translations are in progress. See original paper for figures.