SPECTRAL SYNTHESIS FOR THE SHIFT OPERATOR AND ZEROS IN CERTAIN CLASSES OF ANALYTIC FUNCTIONS, SMOOTH UP TO THE BOUNDARY

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SPECTRAL SYNTHESIS FOR THE SHIFT OPERATOR AND ZEROS IN CERTAIN CLASSES OF ANALYTIC FUNCTIONS, SMOOTH UP TO THE BOUNDARY

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This note considers the problem of spectral synthesis* for invariant subspaces of the shift operator in weighted classes of sequences (or of the corresponding functions). This problem is related to a certain uniqueness (or quasianalyticity) theorem for functions regular in the unit disk. The proof of the latter theorem makes it possible to formulate, under known restrictions, a criterion for the possibility of spectral synthesis.

§ 1. **Notation.**

1.1. $l^p(w) \equiv l^p(w_n^p)$ is the space of all sequences $x = \{x_n\}_{n=0}^{\infty}$ of complex numbers summable with power p, $1 \le p \le \infty$, with weight $\{w_n^p\}_{n=0}^{\infty}$ (bounded, if $p = \infty$), equipped with the usual norm,

$$l^p(w) = l^p, \quad \text{if } w_n \equiv 1,\ n \ge 0.$$

$$l_0^p(w) = \{x \in l^p(w) : x = \{x_n\}_{n=0}^{\infty},\ \lim_n x_n w_n = 0\}, \quad 1 \le p \le \infty.$$

1.2. If $x \in l^p(w)$, then

$$\hat{x}(Z) = \sum_{n=0}^{\infty} x_n Z^n$$

is a formal power series convergent for complex z,

$$|z| < \varliminf_n w_n^{1/n}. \qquad l_A^p(w) = \{\hat{x} : x \in l^p(w)\}.$$

1.3. S is the shift operator in the space of (all) sequences of complex numbers:

$$S(x_0, x_1, ...) = (0, x_0, x_1, ...),$$

or

$$(S\hat{x})(Z) = Z\hat{x}(Z)$$

in the sense of multiplication of formal power series.

1.4. If $\lambda_n = w_{n+1}/w_n,\ n \geq 0$, then $T = S\Lambda$ is the weighted shift operator in the space of (all) sequences, and

$$\Lambda\{x_n\}_{n=0}^{\infty} = \{\lambda_n x_n\}_{n=0}^{\infty}.$$

1.5.

$$r(T) \equiv r_w(S) \overset{\text{def}}{=} \overline{\lim}_n\, w_n^{1/n}.$$

1.6. If f is a function from $l_A^p(w)$, then $N(f)$ is the family of all its zeros (with multiplicities taken into account) lying in

$$D_r = \{z : |z| < r_w(S)\}$$

or in

$$\overline{D}_r = \{z : |z| \leq r_w(S)\},$$

if $l_A^p(w) \subset C(\overline{D}_r)$.

* One says that a linear operator T **admits spectral synthesis** if any of its (closed) invariant subspaces is generated by the root vectors of the operator T contained in it. Together with a simple description of all invariant subspaces of the operator T, the possibility of spectral synthesis leads to effective criteria for completeness of systems of the form $\{T^n x\}_{n=0}^{\infty}$.

§ 2. **Additional assumptions.**

α. $\{\log w_n\}_0^{n=\infty}$ is a concave sequence, $r_w(S) = 1,\ w_n \uparrow$.

β. $n^{-3/2} \log w_n \downarrow 0,\ r_w(S) = 1,\ w_n \uparrow$.

γ. $l_A^p(w)$ is a Banach algebra (with multiplication of formal power series), $r_w(S) = 1,\ w_n \uparrow$.

In addition, throughout this note it is assumed that $w_n > 0$, and

$$\sup_n w_{n+1}/w_n < \infty$$

(the boundedness of the operator S in $l^p(w)$).

§ 3. **Assertions.** The aim of the note is to clarify the connections between the following assertions:

A. $l^p_A(w)$ is contained in $C^\infty(\overline{D}_1)$ and is a quasianalytic class of functions in $\overline{D}_1$.

B. $f \in l^p_A(w)$ and $N(f)$ is infinite $\Rightarrow f \equiv 0$ (uniqueness theorem).

C. S-invariant subspaces in $l^p_A(w)$ are determined by their zeros* (i.e., for S^* in $l^{p'}_0(w_n^{-p'})$ spectral synthesis is possible).

D.

$$\sum_{n=0}^{\infty} \frac{\log w_n}{n^{3/2}+1} = +\infty.$$

§ 4. **Theorems.**

Theorem 1**. $\mathrm{B} \overset{\beta}{\Longrightarrow} \Gamma \overset{\alpha}{\Longleftrightarrow} \mathrm{A} \overset{\beta'}{\Longleftrightarrow} \mathrm{B}$.

Theorem 2. Let the sequence $\{w_n\}_0^\infty$ satisfy condition β, while condition D is not fulfilled for it. Then:

1) For any sequence $\{\lambda_n\}_{n=1}^\infty$, $|\lambda_n| < 1$, $\sum_n (1-|\lambda_n|) < \infty$, lying in the angle

$$\{\xi : \ |\arg(1-\xi)| \le \theta < \pi/2\},$$

there exists f, $f \in l^p_A(w)$, such that $f(\lambda_n) = 0$, $n \ge 1$, $f \not\equiv 0$.

2) There exists an outer $^{(1)}$ function m, continuous in $\overline{D}_1$, $m \in l^p_A(w)$, such that $\psi_\alpha m \in l^p_A(w)$ for every α, $\alpha > 0$,

$$\psi_\alpha(z) = \exp\left(\alpha \, \frac{z+1}{z-1}\right), \quad |z| < 1,$$

and $\psi_\alpha m$, $\alpha > 0$, generate distinct S-invariant subspaces in $l^p(w)$.

Theorem 3. Let the sequence $\{w_n\}_0^\infty$ be such that either $r_w(S) = 0$, or condition β is fulfilled. Then each of the following assertions follows from the preceding one:

1) $T = S\Lambda$ is a basis operator in l^p_0 in the sense of the definition given in (2).

2) $l^p_A((w_{n+j})_{n\ge 0})$ is an algebra for every j, $j \ge 0$, admitting spectral synthesis of ideals.

3) $l^p_A((w_{n+j})_{n\ge 0})$ is an algebra for every j, $j \ge 0$, and either $r(T) = 0$, or $r(T) > 0$ and condition D is fulfilled.

Moreover***, if $r(T) = 0$, then 3) $\Rightarrow$ 1).

Fig. 1

Figure 1: Fig. 1

Lemma. In order that the space $l^p_A(w)$ be a Banach algebra, it is sufficient, and for $p = 1, \infty$ also necessary, that

$$\sup_{n\geq 0}\left(\sum_{k=0}^{n}\frac{w_n^{p'}}{w_k^{p'}w_{n-k}^{p'}}\right)^{1/p'} < \infty, \qquad \frac{1}{p}+\frac{1}{p'}=1.$$

* That is, if $SM \subset M$, M is a closed subspace in $l^p_A(w)$, then

$$M \overset{\text{def}}{=} M_E = \{f \in l^p_A(w) : N(f) \supset E\},$$

where E is some family of complex numbers.

** The proposition A $\Rightarrow$ B means that A implies B under condition β.

*** After the manuscript of the note had been submitted for publication, R. Gellar informed the author in a letter that for a basis operator $T = S\Lambda$ one always has $r(T) = 0$.

Corollary. The operator $T = S\Lambda$ with $r(T) = 0$ is basic in l^p_0 if

$$\sup_{n\geqslant 0}\left(\sum_{k=0}^{n}\frac{w_{n+j}^{p'}}{w_{k+j}^{p'}w_{n-k+j}^{p'}}\right)^{1/p'} < \infty, \quad j = 0, 1, \ldots$$

For $p = 1, \infty$ the converse assertion is also true.

§ 5. **Some remarks.**

5.1. In the case $r_w(S) = 0$, the possibility of spectral synthesis for the operator S^* is equivalent to the single-cell property (linear similarity) of the operators S and T. Therefore Theorem 3 and the corollary give a new strengthening (for $p = \infty$, a final one) of the results on basicity and the single-cell property for operators $T = S\Lambda$ obtained in (^{2}).

5.2. As early as 1952 L. Carleson proved (^{3}) that $\Gamma \overset{\alpha}{\Rightarrow}$ B (the uniqueness theorem: if $N(f)$ is infinite and the conditions α, Γ are fulfilled, then $f \equiv 0$). By virtue of Theorem 1 this assertion means that every S^*-invariant subspace is finite-dimensional. We also note that the presence of spectral synthesis for the operator S^* (for example under the conditions given in Theorem 1) leads to a simple description of the cyclic ($\equiv$ weakly invertible) elements of the operator S: $f \in l^p_A(w)$ is cyclic (i.e. $\{S^n f\}_{n=0}^{\infty}$ is complete in $l^p_A(w)$) $\iff f(\lambda) \neq 0$ for every λ.

Fig. 1

5.3. The existence of continual chains of invariant subspaces corresponding to the "boundary zero" (see Theorem 2, item 2) was discovered by A. Beurling (^{4}) in 1949 for the space l^2. Such chains were later studied in (^{5},^{6}) (for the space $C_A = \{f \in C(\overline{D}_1) : f \text{ is analytic in } D_1\}$) and (^{7},^{8}) (for the algebras $L^1(\varphi; 0, \infty)$ on the half-axis $[0, \infty)$; in particular, (^{8}) gives a complete description of the primary ideals in such algebras in the non-quasianalytic case).

5.4. The assertion $\Gamma \overset{\alpha}{\Longleftrightarrow}$ A is derived from the theorem of R. Salinas (^{9})–M. M. Dzhrbashyan–B. I. Korenblum (^{10})* on the quasianalyticity of the classes

$$D\{A_n\} = \{f \in C(\overline{D}_1) : \|f^{(n)}\| \leqslant K_f A_n,\ n \geqslant 0\}.$$

5.5. For the proof of the inclusion B ⇒ A, certain assumptions on the regularity of the sequence $\{w_n\}_{n=0}^{\infty}$ (for example β) are necessary, whereas A ⇒ B is valid without additional restrictions.

5.6. The main difficulty in the proof of Theorem 1 is the assertions B ⇒ A and B ⇒ Γ, which mean that without condition Γ the spaces $l^p_A(w)$ contain "bad" functions, more precisely, the functions from items 1)–2) of Theorem 2. The estimate of the Taylor coefficients of these functions ($\psi_\alpha m$ and Bm, B being the infinite Blaschke product (^{1})) is carried out by deforming the contour in the integral

$$(\psi_\alpha m)_k = \frac{1}{2\pi i}\int_{\partial D_1} \frac{\psi_\alpha(\zeta)m(\zeta)}{\zeta^{k+1}}\,d\zeta = \frac{1}{2\pi i}\int_{L_k} \frac{\psi_\alpha(\zeta)m(\zeta)}{\zeta^{k+1}}\,d\zeta,$$

where L_k is a curve composed of a part of the circle $\{z : |z| = r_k\}$, $r_k > 1$, and a part of the curve L (see Fig. 1), the order of tangency of which with the circle ∂D_1 is regulated by the condition

$$\sum_0^\infty \frac{\log w_n}{n^{3/2}+1} < \infty.$$

* It should be noted that the theorem mentioned was found in 1955 (independently and almost simultaneously) by the first two authors, and was then rediscovered in (^{10}).

5.7. Condition γ in Theorem 1 can be replaced by a Wiener-type condition for $l^p_A(w)$: if $f \in l^p_A(w)$ and $\hat{f}(\lambda) \neq 0$ for all λ, then f is a weakly invertible element in $l^p_A(w)$.

5.8. The Borel transform (11) maps the space $l^p_A(w)$ into the collection B_p of all entire functions f of exponential type for which $\{f^{(n)}(0)\}_0^\infty \in l^p(w)$. Under this correspondence the operator S corresponds to the integration operator J,

$$(Jf)(z) = \int_0^z f(t)\,dt, \qquad f \in B_p.$$

The description of the cyclic elements of S (see Remarks 1 and 2) now leads to theorems on completeness (and basicity) of systems of successive primitives $\{J^n f\}_{n=0}^{\infty}$ in the spaces B_p (see also (^{2}, ^{12})).

5.9. Weight sequences $\{w_n\}_0^\infty$ with

$$w_n = ecn^\delta, \quad w_n = \exp\bigl(cn^\delta(\log\cdots\log n)^{\varepsilon_1}\bigr),$$

$$w_n = \exp\bigl(cn^\delta(\log\cdots\log n)^{\varepsilon_1}(\log\cdots\log n)^{\varepsilon_2}\bigr)$$

and so on ($c > 0$, $0 \leqslant \delta < 1$, $\varepsilon_i > 0$, s_i are natural numbers) satisfy all the conditions α–γ, and for them all assertions A–Γ are equivalent.

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