On Spectral Expansions Corresponding to an Arbitrary Nonnegative Self-Adjoint Extension of the Laplace Operator

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Abstract

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MATHEMATICS

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On Spectral Expansions Corresponding to an Arbitrary Nonnegative Self-Adjoint Extension of the Laplace Operator

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The paper develops a universal method that makes it possible to study spectral expansions corresponding to an arbitrary nonnegative self-adjoint extension of the Laplace operator in an arbitrary bounded or unbounded domain with any kind of spectrum (point, continuous, or mixed). This method is based on combining the basic apparatus of our investigations (^{1}-^{5}) with the well-known theorem of Gårding (^{6}), and makes it possible, in each of the function classes of Sobolev, Nikol' skii, Besov, Liouville, and Zygmund—Hölder,* to establish definitive conditions for uniform convergence and Riesz summability of the indicated expansions, which are new even for the case of the ordinary N-fold Fourier integral.

Let G be an arbitrary domain in the space E_N, L an arbitrary nonnegative self-adjoint extension of the operator $-\Delta$ in the domain G, $\{E_\lambda\}$ the spectral family of the indicated extension, and $\theta(x,y,\lambda)$ the kernel of E_λ, called the spectral function of the extension L.** Then, for any function $f(x) \in L_2(G)$,

$$E_\lambda f(x) = S_\lambda(x,f) = \int_G \theta(x,y,\lambda) f(y)\, dy. \tag{1}$$

The Riesz means of order s of the spectral expansion (1) for any function $f(x) \in L_2(G)$ are defined by the equality

$$\sigma_\lambda^s(x) = \sigma_\lambda^s(x,f) = \int_0^\lambda \left(1 - \frac{t}{\lambda}\right)^s dS_t(x,f). \tag{2}$$

In the present paper we study the Riesz means (2) of any order s satisfying the condition $0 \le s < (N-1)/2$.***

We begin by determining conditions that do not ensure not only uniform convergence but even localization of the Riesz means of order $0 \le s < (N-1)/2$.

Theorem 1 (on conditions that do not ensure localization of Riesz means of order s in the Zygmund—Hölder classes). *Let* $N \geq 2$*, let* G *be an arbitrary* N*-dimensional domain, let* L *be an arbitrary nonnegative self-adjoint extension of the operator* $-\Delta$ *in the domain* G*, let* x_0 *be any interior point of* G*, and let* a *be any fixed number satisfying the inequalities* $0 < a < (N-1)/2 - s$*. Then there exists a function* $f(x)$ *satisfying the following requirements: 1)* $f(x)$ *is finite in the domain* G *and vanishes in some neighborhood* D *of the point* x_0*; 2)* $f(x) \in C^a(G)$*;*

* The definition of the indicated classes can be found in (^{7}).

** For the properties of $\theta(x, y, \lambda)$, see (^{8}).

*** For $s = 0$ the Riesz means (2) pass into the spectral expansion (1) itself, which is thereby included in our consideration. The case of Riesz means of order $s \geq (N-1)/2$ will be studied in a separate paper.

3) the Riesz means (2) of the spectral expansion of $f(x)$ have no limit at the point x_0 as $\lambda \to \infty$.

Corollary 1 (on conditions that do not ensure localization of Riesz means of order s in the classes H_p^α, $B_{p,\theta}^\alpha$, L_p^α, and W_p^α). *If* N*,* G*,* x_0*, and* α *have the same meaning as in Theorem 1, then there exists a function* $f(x)$ *satisfying the following requirements: 1)* $f(x)$ *is finite in the domain* G *and vanishes in some neighborhood* D *of the point* x_0*; 2)* $f(x)$ *belongs in the domain* G *to each of the classes* H_p^α*,* $B_{p,\theta}^\alpha$*,* L_p^α*, and* W_p^α *for all* $p \geq 1$*,* $\theta \geq 1$*; 3) the Riesz means (2) of the spectral expansion of* $f(x)$ *have no limit at the point* x_0 *as* $\lambda \to \infty$*.*

Thus, membership of the function $f(x)$ in any of the indicated five classes with order of differentiability α less than $(N-1)/2 - s$ does not ensure even localization of the Riesz means (2) (whatever the degree of summability p may be, and, in the case of the Besov class, for $\theta \geq 1$).

The question naturally arises of studying the Riesz means (2) for functions belonging to the above-mentioned five classes with order of differentiability $\alpha \geq (N-1)/2 - s$.

Let us agree to denote by the symbol G_h the set of points of the domain G whose distance from the boundary of G is not less than the number $h > 0$.*

Theorem 2 (on conditions ensuring uniform convergence of the Riesz means of order s in Nikol' skii classes). *Let $N \geq 2$, let G be an arbitrary N-dimensional domain, let L be an arbitrary nonnegative self-adjoint extension of the operator $-\Delta$ in the domain G, and let $f(x)$ be an arbitrary function satisfying the following requirements: 1) $f(x)$ vanishes outside the set G_{h_0} for some $h_0 > 0$; 2) $f(x)$ belongs throughout the domain G to the class H_p^α with $\alpha \geq (N-1)/2 - s$; 3) in some domain D** contained in G, the function $f(x)$ belongs to the class H_p^α for all α and p satisfying the conditions $\alpha \geq (N-1)/2 - s$, $p\alpha > N$, $p \geq 1$. Then, for every $h > 0$, the Riesz means (2) of the spectral expansion of $f(x)$ converge to $f(x)$ as $\lambda \to \infty$ uniformly on the set D_h.*

Corollary 2. *Since the class* H_p^α *contains any of the classes* $B_{p,\theta}^\alpha$, L_p^α, W_p^α, *and* C^α, *in the formulation of Theorem 2, instead of the class* H_p^α *one may take any of the classes* $B_{p,\theta}^\alpha$, L_p^α, W_p^α, *and* C^α, *with the same order of differentiability* α, *with the same degree of summability* p, *and (for the case of the class* $B_{p,\theta}^\alpha$*) with any* $\theta \geq 1$.

Thus, in each of the five classes of functions studied, we have established the definitive order of differentiability $\alpha \geq (N-1)/2 - s$, which ensures uniform convergence of the Riesz means of order s (for $\alpha < (N-1)/2 - s$, localization of the Riesz means of order s is not ensured in any of the indicated classes). Moreover, the degree of summability p found by us, satisfying the inequality $p\alpha > N$, is also definitive, since in any of the classes $B_{p,\theta}^\alpha$, L_p^α, and W_p^α for $p > 1$, $\theta \geq 1$, and in the class H_p^α even for $p \geq 1$, the inequality $p\alpha \leq N$ permits the existence of an unbounded function for which the Riesz means (2) certainly do not converge uniformly.

The proof of the formulated results is based on combining the apparatus developed for the point spectrum in ($^{1-5}$) with the fundamental theorem of Gårding (^{6}) (see also (^{9}), p. 875).

Denote by U an arbitrary ordered representation of the space $L_2(G)$ relative to the self-adjoint extension L, with measure μ, multiplicity sets e_k, kernels $w_k(x, \lambda)$, and multiplicity $m \leq \infty$. Then

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* In the case when the domain G coincides with all of E_N, the set G_h coincides with G for every $h > 0$. In this case there exists a unique nonnegative self-adjoint extension of the operator $-\Delta$, leading to the expansion in the ordinary N-fold Fourier integral.

** Of course, the domain D may coincide with G.

the Riesz means (2) of any function $f(x) \in L_2(G)$ have the form

$$\sigma_\lambda^s(x, f) = \sum_{k=1}^{m} \int_0^\lambda \left(1 - \frac{t}{\lambda}\right)^s (Uf)_k(t) w_k(x, t)\, d\mu(t),$$

where the Fourier transform $(Uf)_k(\lambda)$ of the function $f(x)$ is defined by the equality

$$(Uf)_k(\lambda) = \int_G f(x) w_k(x, \lambda)\, dx.$$

For the application of the method developed in ($^{1-5}$), it is essential only that the kernels $w_k(x, \lambda)$ are regular solutions of the equation $\Delta w_k + \lambda w_k = 0$ and that, for any $f(x)$ and $g(x)$ in the class $L_2(G)$, the Parseval equality

$$\int_G f(x)g(x)\,dx = \sum_{k=1}^{m} \lim_{\lambda\to\infty} \int_0^{\lambda} (Uf)_k(t)(Ug)_k(t)\,d\mu(t)$$

holds.

Let us note the most important lemmas.

Lemma 1.

$$\sum_{k=1}^{m} \int_{\mu^2}^{(\mu+1)^2} |w_k(x,\lambda)|^2\,d\mu(\lambda) = O(\mu^{N-1})$$

for any $\mu > 0$, uniformly with respect to $x \in G_h$ for any $h > 0$.

Lemma 2. For any interior point x_0 of the domain G and any $s \geqslant 0$, there exists an open set E, contained inside G, not containing the point x_0, and such that, for some $\beta > 0$,

$$\int_E \left| \sum_{k=1}^{m} \int_0^{\lambda} \left(1 - \frac{t}{\lambda}\right)^s w_k(x_0,t) w_k(y,t)\,d\mu(t) \right| dy \geqslant \beta \cdot \lambda^{(N-1)/4 - s/2}.$$

Lemma 3. Let $s > 0$, $\beta > 0$, and let $u(\lambda)$ be an arbitrary function integrable with respect to the measure $\mu(\lambda)$, $s = r + \chi$, where r is an integer, $0 < \chi \leqslant 1$,

$$\sigma_\lambda^s = \int_0^{\lambda} u(t)\left(1 - \frac{t}{\lambda}\right)^s d\mu(t), \qquad \overline{\sigma}_\lambda^s = \int_0^{\lambda} u(t)t^{\beta}\left(1 - \frac{t}{\lambda}\right)^s d\mu(t).$$

Then the equality holds

$$\overline{\sigma}_\lambda^s = \lambda^{\beta}\sigma_\lambda^s + (-1)^{r+1}\lambda^{-s} \int_0^{\lambda} \frac{d^{r+2}}{dt^{r+2}} \left[(\lambda - t)^s (t^{\beta} - \lambda^{\beta})\right] \frac{t^{r+1}}{(r+1)!} \sigma_t^{r+1}\,dt.$$

We shall agree to call a function $T_\alpha(x,y)$ a **kernel of order** α if, for any point y fixed inside the domain G, it has a Fourier transform $(UT_\alpha)_k(y,\lambda)$ equal to $w_k(y,\lambda)(1+\lambda)^{-\alpha/2}$.

Lemma 4. Let h be any positive number, y any point of the set G_h, n any number, $r = r_{xy}$ the distance between the points x and y, and $K_\nu(x)$ the so-called Macdonald function of order ν. Then the kernel $T_\alpha(x,y)$ of any order $\alpha > 0$ is representable in the form $T_\alpha(x,y) = \varphi_\alpha(x,y) + \Psi_\alpha(x,y)$, where

$$\varphi_\alpha(x,y) = \overline{\varphi}_\alpha(r) = \begin{cases} 2\left[(2\pi)^{N/2}\Gamma\left(\frac{\alpha}{2}\right)\sqrt{2^\alpha}\right]^{-1} r^{\frac{\alpha-N}{2}} \cdot K_{\frac{N-\alpha}{2}}(r) + \sum\limits_{k=0}^{n} a_k r^{2k}, & \text{if } r \leqslant h, \\ 0, & \text{if } r > h, \end{cases}$$

and the constants a_k $(k = 0, 1, \ldots, n)$ are chosen so that, at $r = h$, the function $\overline{\varphi}(r)$ and all its derivatives up to order n, inclusive, turn-

vanishes at zero,* and the function $\Psi_a(x,y)$ has the form

$$\Psi_a(x,y) = \sum_{k=1}^{n} \lim_{\lambda\to\infty} \int_0^\lambda w_k(x,t) w_k(y,t)\, \gamma_k(t)\, d\mu(t),$$

where $\gamma_k(t) = O\left[(1+t)^{-N/4-n/2}\right]$.

Lemma 5. If, for $\alpha > 0$ and $h > 0$, the function $f(x)$ belongs to the class $\overset{\circ}{H}{}^\alpha_2(G_h)$, where $\overset{\circ}{H}{}^\alpha_p(G_h)$ is the class of functions obtained by completing, in the metric H^α_p, the set of functions finite and infinitely differentiable in G_h, then for any $\lambda > 0$

$$\sum_{k=1}^{m} \int_\lambda^{2\lambda} |(Uf)_k(t)|^2 t^\alpha\, d\mu(t) \le C_h \|f\|_{\overset{\circ}{H}{}^\alpha_2}.$$

Lemma 6. Suppose that, for some domain D contained in G, the function $f(x) \in H^\alpha_p(D)$ for $p \ge 1$, $\alpha p > N$, $\alpha = l + \chi$, where $0 < \chi \le 1$, $l = 0, 1, \ldots$. Further, let $\psi(r) = \psi(r,x) = \omega_N^{-1} \int_\theta f(x + r\theta)\, d\theta$ be the mean value of the function f on the surface of the sphere of radius r with center at the point x, and let $\varphi_m(r) = r^{m+\chi-1}\psi^{(m)}(r)$, where $m = 0, 1, \ldots, l$. Then for any ρ from the interval $0 < \rho < h$, uniformly with respect to x from the set D_{2h}, the estimate

$$\int_0^h |\varphi_m(r+2\rho) - 2\varphi_m(r+\rho) + \varphi_m(r)|\, dr \le C_h \rho^\chi \|f\|_{H^\alpha_p}$$

holds.

Lemma 7. Let $F(r) = r^{N-1}\psi(r,x) = r^{N-1}\omega_N^{-1}\int_\theta f(x + r\theta)\, d\theta$, $J_\nu(t)$ be the Bessel function of order ν, $v_\nu(t) = \sqrt{t}\, J_\nu(t)$, $DF = \dfrac{d}{dr}\left[\dfrac{F(r)}{r}\right]$, $D^{kF} = D(D^{k-1}F)$. Then, if $f(x) \in \overset{\circ}{H}{}^\alpha_2(G_h)$ for $\alpha \ge (N-1)/2 - s$, $\alpha = l + \chi$, where l is an integer, $0 < \chi \le 1$, then for $\nu = N/2 + s - l$, uniformly with respect to x on the set G_{2h}, the estimate

$$\left|\sigma_\lambda^s(x) - 2^s\Gamma(s+1)2^{N/2-1}\Gamma\left(\frac{N}{2}\right)\lambda^{\chi/2}\int_0^h v_\nu(r\sqrt{\lambda})\, r^{2l-N+\chi} D^{lF}\, dr\right| \leq$$

$$\leq C_h \|f\|_{\mathring{H}_2^\alpha(G_h)}$$

holds.

Lemmas 1-4 and Theorem 1 were proved by V. A. Il' in, and Lemmas 5-7 and Theorem 2 by Sh. A. Alimov.

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* These conditions determine the constants a_k $(k = 0, 1, \ldots, n)$ uniquely.

Note: Figure translations are in progress. See original paper for figures.

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