THE HYDROGEN ATOM AND THE CONFORMAL GROUP

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The purpose of the present note is to express, in the most direct way, the Schrödinger equation for the hydrogen atom in terms of the generators L_{ab} of the conformal group, satisfying the commutation relations:

$$\begin{gathered} a, b = 0,1,2,3;\ 5,6, \\ g_{ab} = +,-,-,-;\ -+, \\ [L_{ab}, L_{cd}] = i(g_{ad}L_{bc} + g_{bc}L_{ad} - g_{ac}L_{bd} - g_{bd}L_{ac}). \end{gathered} \tag{1}$$

The 15 generators L_{ab} include, besides the generators $M_{\mu\nu} = L_{\mu\nu}$ (the Lorentz group), $\mu, \nu = 0,1,2,3$, another 9 generators $L_{\mu 5}, L_{\mu 6}, L_{56}$, from which are formed the translation operators Π_μ, the special conformal transformation K_μ, and the dilatation D:

$$\Pi_\mu = L_{\mu 6} + L_{\mu 5}, \qquad K_\mu = L_{\mu 6} - L_{\mu 5}, \qquad D = L_{56}. \tag{2}$$

The transformations are carried out in three stages.

1. We pass to the Heisenberg representation

$$L_{ab}(\Pi, y) = \exp(-iy_\sigma \Pi_\sigma) L_{ab} \exp(iy_\sigma \Pi_\sigma),$$

where $y = (y_0, \mathbf{y})$ are coordinates, and expand $L_{ab}(\Pi, y)$ in a Maclaurin series

$$L_{ab}(\Pi, y) = L_{ab} + (-i)y_\nu[\Pi_\nu, L_{ab}] + \frac{1}{2}(-i)^2 y_\mu y_\nu [\Pi_\mu[\Pi_\nu L_{ab}]] + \cdots. \tag{3}$$

Using formulas (1) and (2), we calculate

$$[\Pi_\sigma, M_{\mu\nu}] = i(g_{\sigma\mu}\Pi_\nu - g_{\sigma\nu}\Pi_\mu), \qquad [\Pi_\sigma, D] = -i\Pi_\sigma,$$
$$[\Pi_\sigma, K_\mu] = -2i(g_{\sigma\mu}D + M_{\sigma\mu}), \tag{4}$$
$$[\Pi_\tau, [\Pi_\sigma, K_\mu]] = 2(g_{\tau\sigma}\Pi_\mu - g_{\sigma\mu}\Pi_\tau - g_{\tau\mu}\Pi_\sigma),$$

and from formula (3) obtain

$$M_{\mu\nu}(\Pi, y) = M_{\mu\nu} + y_\mu\Pi_\nu - y_\nu\Pi_\mu,$$
$$K_\mu(\Pi, y) = K_\mu + 2y_\sigma(M_{\mu\sigma} - g_{\mu\sigma}D) + 2y_\mu(y_\sigma\Pi_\sigma) - (y_\sigma y_\sigma)\Pi_\mu, \tag{5}$$
$$D(\Pi, y) = D - y_\sigma\Pi_\sigma.$$

As the expression for $M_{\mu\nu}(\Pi, y)$ shows, expansion (3) generalizes the well-known decomposition of the total angular momentum into spin and orbital parts.

We are especially interested in the generator $K_0(\Pi, y)$ (acting on the scalar basis consisting of solutions $\varphi(y)$ of the d' Alembert equation, decomposed into harmonics of positive frequency). From (5) we have

$$K_0(\Pi, y)\varphi(y) = [2y_0(y_\sigma\Pi_\sigma) - (y_\sigma y_\sigma)\Pi_0]\,\varphi(y). \tag{6}$$

2. We now perform the canonical transformation:

$$y \to -P, \qquad \Pi \to x, \qquad \varphi(y) \to \psi(x), \tag{7}$$

imposing on the new coordinates x and the new basis $\psi(x)$ the following conditions:

$$(x_\sigma x_\sigma) = 0, \qquad P_0\psi(x) = -y_0\psi(x) = 0, \qquad \Pi_0\psi(x) = x_0\psi(x) = r\psi(x), \tag{8}$$

where $r = \sqrt{(\mathbf{xx})}$, $\psi(x) = \tilde{\varphi}(r, \mathbf{x})$, and the sign $\sim$ denotes the Fourier transform:

$$\varphi(y) = \int \frac{\tilde{\varphi}(r, \mathbf{x})}{2r}\, e^{i(y_0 r - \mathbf{yx})}\, d^3x. \tag{9}$$

We obtain, by virtue of (8),

$$\tilde{K}_0(P, x)\psi(x) = \{2(x_0P_0 - (\mathbf{xP}))P_0 - x_0(P_0^2 - \mathbf{P}^2)\}\psi(x) = r\mathbf{P}^2\psi(x),$$

$$\tilde{\Pi}_0(x)\psi(x) = r\psi(x). \tag{10}$$

It is easy to see that, since $P_0\psi(x) = 0$, the order of the factors in (10) is immaterial. The operators K_0 and Π_0, transferred to the space $\psi(x)$, are denoted by $\tilde{K}_0$ and $\tilde{\Pi}_0$. These operators are Hermitian if the metric is defined as

$$(\psi(x), \psi(x)) = \frac{\psi^*(x)\psi(x)}{r}\, d^3x.$$

3. Multiplying the Schrödinger equation

$$\left(\frac{1}{2m}\mathbf{P}^2 - \frac{e^2}{r} - E\right)\psi(x) = 0$$

by $2mr$, we bring it to the form

$$\{r\mathbf{P}^2 - (2mE)r - 2me^2\}\psi(x) = 0. \tag{11}$$

Substituting the expressions (10) into (11), we obtain

$$\{\tilde{K}(P, x) - 2mE\tilde{\Pi}(x) - 2me^2\}\psi(x) = 0 \tag{11a}$$

or, using (2) and omitting the arguments of $\tilde{L}_{ab}(P, x)$,

$$\{\tilde{L}_{06}(1 - 2mE) - \tilde{L}_{05}(1 + 2mE) - 2me^2\}\psi = 0, \tag{11b}$$

which solves the problem posed. Equation (11b) is the dynamical equation for the hydrogen atom in the theory of the conformal group, corresponding to the Schrödinger equation. To compute the eigenvalues, following works ([1,2]), we perform the canonical transformation $\tilde{L}_{ab} \to \tilde{\tilde{L}}_{ab}$ (the so-called "tilt" —a hyperbolic rotation in the 5-6 plane)

$$\tilde{L}_{06} = \tilde{\tilde{L}}_{06}\operatorname{ch}\xi + \tilde{\tilde{L}}_{05}\operatorname{sh}\xi, \qquad \tilde{L}_{05} = \tilde{\tilde{L}}_{06}\operatorname{sh}\xi + \tilde{\tilde{L}}_{05}\operatorname{ch}\xi.$$

Substituting into (11b), we obtain

$$\{\tilde{\tilde{L}}_{06}[(1 - 2mE)\operatorname{ch}\xi - (1 + 2mE)\operatorname{sh}\xi] - 2me^2 -$$

$$-\tilde{\tilde{L}}_{05}[(1 - 2mE)\operatorname{sh}\xi - (1 + 2mE)\operatorname{ch}\xi]\}\psi = 0. \tag{12}$$

For

$$\operatorname{th}\xi = \frac{1 + 2mE}{1 - 2mE}$$

the coefficient of $\tilde{\tilde{L}}_{05}$ vanishes, and we obtain

$$\{\tilde{\tilde{L}}_{06}[(1-2mE)^2-(1+2mE)^2]^{1/2}-2me^2\}\psi=0.$$

The compact generator $\tilde{\tilde{L}}_{06}$ has discrete eigenvalues $n=1,2,3,...$, whence we obtain the discrete spectrum

$$E=-\frac{1}{2n^2}m^2e^4 \qquad (E<0,\ \hbar=1).$$

For $\operatorname{th}\xi=(1-2mE)/(1+2mE)$, the coefficient of L_{06} vanishes, and we obtain

$$\left\{\widetilde{L}_{05}\left[(1-2mE)^2-(1+2mE)^2\right]^{1/2}+2me^2\right\}\psi=0.$$

The noncompact generator $\widetilde{L}_{05}$ has the continuous spectrum iv, where $0<v<\infty$, whence we obtain

$$E=\frac{1}{2v}m^2e^4 \qquad (E>0,\ \hbar=1).$$

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Note: Figure translations are in progress. See original paper for figures.