ON THE LOCAL STRUCTURE OF NORMAL FIELDS

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Abstract

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ON THE LOCAL STRUCTURE OF NORMAL FIELDS

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Let T be a separable compact topological space with the second axiom of countability; (Ω, B, P) the basic probability space; x_t, $t \in T$, a separable normal field whose correlation function $R(t,s)$ is continuous: $R : T \otimes T \to R^1$. It is also assumed that on T there exists a finite measure μ, regular with respect to the topology T; we shall denote the topology by τ: $\mu(T) = 1$. C is the space of all continuous functions mapping T into R^1, with norm $\forall f \in C$

$$\|f\| = \sup_{t \in T} |f(t)|;$$

L_2 is the space of functions quadratically summable with respect to μ, with norm

$$\|f\|^2_{L_2} = \int_T f^2(t)\, d\mu(t).$$

Denote by λ_k and $\varphi_k(t)$, respectively, the eigenvalues and eigenfunctions of the operator $R : L_2 \to L_2$, given by the formula

$$(Rf)(t) = \int_T f(s) R(t,s)\, d\mu(s),$$

where $\|\varphi_n\|_{L_2} = 1$; by the theorems of Mercer and Hilbert-Schmidt,

$$\varphi_k \in C; \qquad \lambda_k \geq 0; \qquad (k = 1, 2, ...),$$

$$R(t,s) = \sum_{k=1}^{\infty} \lambda_k \varphi_k(t) \varphi_k(s), \tag{1}$$

where the series in (1) converges absolutely and uniformly (see ([2]), p. 250). Put

$$\xi_k = \frac{1}{\sqrt{\lambda_k}} \int_T x_t \varphi_k(t)\, d\mu(t), \tag{2}$$

ξ_k are independent, normal with parameters $(0, 1)$. Denote

$$S_n(t) = \sum_{k=1}^{n} \sqrt{\lambda_k} \xi_k \varphi_k(t).$$

Theorem 1.
$$P\left\{\forall t \in T,\ \lim_{n\to\infty} S_n(t) = x_t\right\} = 1. \tag{3}$$

Theorem 2. The set of discontinuity points of x_t is one and the same for almost all trajectories ([3]).

Theorem 3. In any neighborhood of a discontinuity point, with probability 1, the trajectory is unbounded.

The proof essentially repeats the analogous proof of Yu. K. Belyaev (([1]), pp. 23-33), if instead of the spectral expansion one uses Theorem 1 and the following lemma.

Lemma 1. Let t_0 be a discontinuity point of x_t. Then there exists a nonrandom constant $a > 0$ such that

$$P\left\{\overline{\lim_{t\to t_0}} |x_t - x_{t_0}| > 2a\right\} = 1.$$

Proof. By Theorem 2 the process x_t is discontinuous at t_0 with probability 1; this means that

$$P\left\{\overline{\lim_{t\to t_0}} |x_t - x_{t_0}| = \varepsilon\right\} = 1, \qquad \varepsilon > 0,$$

but then the quantity $\varepsilon = \varepsilon(\omega)$ is measurable with respect to the smallest σ-algebra generated by the quantities $\xi_k, \xi_{k+n}, \dots,\ \forall n \geq 1$, since $S_\xi(t)$ is continuous with probability 1; hence there exists a constant $a > 0$ such that $P\{\varepsilon = 2a\} = 1$, as was required to prove.

Corollary. Let G be a subgroup of the group of homeomorphisms of T onto itself such that for all $g \in G$

$$R(tg, t_1 g) = R(t, t_1).$$

Then the set of discontinuity points of x_t is invariant with respect to G. If, moreover, T is full with respect to G, then the process x_t, if it is discontinuous, is unbounded in every open set.

To derive a necessary and sufficient condition for continuity, suppose that $T = [0, 2\pi]$; without essential loss of generality, $x_0 = x_{2\pi}$. Let $f(x) \in C$ and be periodic; put, for $p \geq 1$, p an integer,

$$\omega_p(f, \delta) = \sup_{|h|<\delta} \sup_x \left| \sum_{k=0}^{p} (-1)^{p-k} C_p^k f(x + kh) \right|,$$

$$K_n(x) = \frac{3}{2\pi n(2n^2+1)} \sin^4\left(\frac{nx}{2}\right) \Big/ \sin^4\left(\frac{x}{2}\right)$$

(the Jackson kernel ([4]), pp. 140-145)

$$K_{mn}(x) = K_m(x) - K_n(x);$$

$$L^k_{mn}(t) = \int_{-\pi}^{\pi} e^{isk} K_{mn}(t-s)\, ds;$$

C^{2m+1} will denote the $(2m+1)$-dimensional complex-number space with coordinates $\forall z \in C^{2m+1}$

$$z = (z_{-m}; z_{-m+1}; \dots z_{-1}; z_0; z_1; \dots z_m),$$

and define

$$(z,z) = \sum_{q=-m}^{m} |z_q|^2; \qquad dz = \prod_{q=-m}^{m} d\,\mathrm{Re}\, z_q\, d\,\mathrm{Im}\, z_q.$$

The Bernstein function $\Phi(z)$ is defined for $z \in C^{2m+1}$ by the equality (see ([5]), p. 127 ff.)

$$\Phi(z) = \sup_x \left| \sum_{k=-m}^{m} z_k e^{ikx} \right|.$$

$\tilde{R}^n_m$ is a square matrix of order $(2m+1; 2m+1)$ with elements

$$(\tilde{R}^n_m)_{kl} = \int_0^{2\pi} \int_0^{2\pi} L^k_{mn}(t)\, \overline{L^l_{mn}(s)}\, R(t,s)\, dt\, ds$$

and R^n_m is the Hermitian square root of $\tilde{R}^n_m$. Also put, for $m > n$,

$$\tau^n_m = (2\pi)^{-2m} \int_{C^{2m+1}} \mathrm{arctg}\, \Phi(R^n_m z) e^{-1/2(z,z)}\, dz.$$

From known theorems of constructive function theory one can derive the following result ([4], pp. 144-145).

Theorem 4. *In order that x_t be continuous with probability 1, it is necessary and sufficient that*

$$\lim_{n\to\infty} \sup_{m>n} \tau_m^n = 0.$$

Theorem 5. *Suppose that the condition of Theorem 4 is satisfied; then there exist constants C_p, d_p such that $\forall p \geqslant 1$*

$$C_p \sup_{m>n} \tau_m^n \leqslant M \operatorname{arctg} \omega_p \left(x_t, \frac{1}{n}\right) \leqslant \frac{d_p}{n^p} \sum_{\nu=1}^{n} \nu^{p-1} \sup_{m>\nu} \tau_m^\nu.$$

By the indicated method one easily obtains the known sufficient conditions for continuity of the process x_t, if one uses the known Bernstein inequality ([4], p. 146).

Put

$$\tilde{\omega}_R(\delta) = \sup_{\substack{|h_1|\leqslant\delta \\ |h_2|\leqslant\delta}} \sqrt{|R(t+h_1, s+h_2) - R(t,s)|}.$$

Theorem 6 (Fernique, [6]). *If*

$$\int_0^\infty \tilde{\omega}_R(e^{-x^2})\, dx < \infty,$$

then x_t is continuous with probability 1.

Corollary. *Let $\omega_1(\delta)$ be a modulus of continuity such that*

$$\overline{\lim_{\delta\downarrow 0}} \frac{\omega_1(\delta)}{\delta} = \infty.$$

Then there exists a stationary process x_t such that for its correlation function $r(t) = Mx_s x_{s+t}$, $\omega_1[r(t), \delta] = \omega_1(\delta)$, but at the same time $P\{x_t \in G\} = 1$; and even $M \exp[\varepsilon \|x_t\|^2] < \infty$ for some $\varepsilon > 0$.

Theorems 4 and 5 have also been obtained for finite-dimensional fields.

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Note: Figure translations are in progress. See original paper for figures.