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MATHEMATICS

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Abstract

Full Text

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MATHEMATICS

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ON THE QUESTION OF GENERALIZING THE INTEGRAL FORMULAS OF CAUCHY, SCHWARZ, AND POISSON

(Presented by Academician M. A. Lavrent′ev on March 6, 1970)

M. M. Dzhrbashyan (^{1}) established generalized formulas of Cauchy, Schwarz, and Poisson associated with a given function $\omega(x) \in \Omega$. Subsequently the author (^{2}) obtained generalized formulas of Cauchy, Schwarz, and Poisson associated with a given system of functions $\omega_j(x) \in \Omega$ $(j = 1, \dots, m)$. In the present note a substantial generalization of these latter formulas is given (Theorems 1 and 2).

1. Let the functions $\omega_j(x) \in \Omega$ $(j = 1, 2, \dots, m)$, $\tilde{\omega}_{\tilde{j}}(x) \in \Omega$ $(\tilde{j} = 1, 2, \dots, \tilde{m})$. Suppose, further, that

$$p_j(0) = 1, \qquad \tilde{p}_{\tilde{j}}(0) = 1,$$

$$p_j(r) = r \int_r^1 \frac{\omega_j(x)}{x^2}\, dx, \qquad \tilde{p}_{\tilde{j}}(r) = r \int_r^1 \frac{\tilde{\omega}_{\tilde{j}}(x)}{x^2}\, dx \quad (r \in (0, 1)),$$

$$\Delta_0^{(j)} = 1, \qquad \widetilde{\Delta}_0^{(\tilde{j})} = 1, \qquad \Delta_k^{(j)} = -(k+1) \int_0^1 r^k \, dp_j(r) = k \int_0^1 r^{k-1} \omega_j(r)\, dr,$$

$$\widetilde{\Delta}_k^{(\tilde{j})} = -(k+1) \int_0^1 r^k \, d\tilde{p}_{\tilde{j}}(r) = k \int_0^1 r^{k-1} \tilde{\omega}_{\tilde{j}}(r)\, dr$$

$$(j = 1, 2, \dots, m; \quad \tilde{j} = 1, 2, \dots, \tilde{m}), \qquad k = 1, 2, \dots \ .*$$

2. In (^{2}) the author introduced into consideration the function

$$C(z; \omega_1, \dots, \omega_m) = \sum_{k=0}^{\infty} \frac{z^k}{\Delta_k^{(1)} \cdots \Delta_k^{(m)}},$$

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which is holomorphic in the disk $|z| < 1$. Therefore, by virtue of Theorem 1 of (^{2}), the function

$$L_{(\widetilde{\omega}1,\dots,\widetilde{\omega}_{\tilde{m}})}\left[C(re^{i\varphi};\omega_1,\dots,\omega_m)\right] \equiv C_{(\widetilde{\omega}1,\dots,\widetilde{\omega}_{\tilde{m}})}(re^{i\varphi};\omega_1,\dots,\omega_m) =$$

$$= \sum_{k=0}^{\infty} \frac{\widetilde{\Delta}_k^{(1)} \cdots \widetilde{\Delta}_k^{(\tilde{m})}}{\Delta_k^{(1)} \cdots \Delta_k^{(m)}} (re^{i\varphi})^k, \tag{1}$$

which, for brevity, we shall denote by $C_{(\widetilde{\omega})}(re^{i\varphi};\omega)$ ($\omega = (\omega_1,\dots,\omega_m)$, $\tilde{\omega} = (\tilde{\omega}_1,\dots,\tilde{\omega}_{\widetilde{m}})$, and similarly everywhere below), is holomorphic in the disk $|z| < 1$.

Introduce the function

$$S_{(\widetilde{\omega})}(z;\omega) = 2C_{(\widetilde{\omega})}(z;\omega) - C_{(\widetilde{\omega})}(0;\omega) = 1 + 2\sum_{k=1}^{\infty} \frac{\widetilde{\Delta}_k^{(1)} \cdots \widetilde{\Delta}_k^{(\widetilde{m})}}{\Delta_k^{(1)} \cdots \Delta_k^{(m)}} z^k, \tag{2}$$

noting that

$$C_{(\widetilde{\omega})}(0;\omega) := \widetilde{\Delta}_0^{(1)} \cdots \widetilde{\Delta}_0^{(\widetilde{m})} / \Delta_0^{(1)} \cdots \Delta_0^{(m)} = 1.$$

Let the function

$$f(re^{i\varphi}) = \sum_{k=0}^{\infty} a_k (re^{i\varphi})^k$$

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* In (^{1}) the function

$$p(r) = r\int_r^1 \frac{\omega(x)}{x^2}\,dx \quad (\omega(x) \in \Omega), \quad r \in (0,1), \quad p(0) = 1,$$

and the sequence of numbers

$$\Delta_k = -(k+1)\int_0^1 r^k\,dp(r) \quad (k = 0,1,2,\dots)$$

were introduced; moreover, it was shown that all the numbers Δ_k $(k = 0,1,2,\dots)$ are positive, with

$$\Delta_0 = 1, \qquad \Delta_k = k\int_0^1 \omega(x)x^{k-1}\,dx \quad (k = 1,2,\dots).$$

be holomorphic in the disk $|z| < R$. Then, by ((2), Theorem 1), the function

$$L^{(\omega_1,\dots,\omega_m)}[f(re^{i\varphi})] \equiv f_{(\omega_1,\dots,\omega_m)}(re^{i\varphi}) = \sum_{k=0}^{\infty} \Delta_k^{(1)} \cdots \Delta_k^{(m)} a_k (re^{i\varphi})^k$$

is holomorphic in the same disk $|z| < R$. Hence it follows that the function

$$f_{(\omega)}(z;\tilde{\omega}) := f_{(\omega_1,\dots,\omega_m)}(z;\tilde{\omega}_1,\dots,\tilde{\omega}_{\widetilde{m}}) = \sum_{k=0}^{\infty} \frac{\Delta_k^{(1)} \cdots \Delta_k^{(m)}}{\widetilde{\Delta}_k^{(1)} \cdots \widetilde{\Delta}_k^{(\widetilde{m})}} a_k z^k \tag{3}$$

is also holomorphic in the disk $|z| < R^*$. Finally, relying on the expansions (1) –(3), we arrive at the following theorem.

Theorem 1. If the function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is holomorphic in the disk $|z| < R$, then for any ρ $(0 < \rho < R)$ the integral formulas

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} C_{(\widetilde{\omega})} \left(e^{-i\theta} \frac{z}{\rho}; \omega \right) f_{(\omega)}(\rho e^{i\theta}; \tilde{\omega})\, d\theta \qquad (|z| < \rho),$$

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_0^{2\pi} S_{(\widetilde{\omega})} \left(e^{-i\theta} \frac{z}{\rho}; \omega \right) \operatorname{Re} f_{(\omega)}(\rho e^{i\theta}; \tilde{\omega})\, d\theta \qquad (|z| < \rho).$$

3. Introduce the function

$$P_{(\widetilde{\omega})}(\theta, r; \omega) = \operatorname{Re} S_{(\widetilde{\omega})}(re^{i\theta}; \omega) = 1 + 2 \sum_{k=1}^{\infty} \frac{\widetilde{\Delta}_k^{(1)} \cdots \widetilde{\Delta}_k^{(\widetilde{m})}}{\Delta_k^{(1)} \cdots \Delta_k^{(m)}} r^k \cos k\theta,$$

which is harmonic in the unit disk $0 \le r < 1,\ 0 \le \theta \le 2\pi$.

Let the function

$$u(re^{i\varphi}) = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos k\varphi - \beta_k \sin k\varphi) r^k$$

$(\alpha_0, \alpha_k, \beta_k$ $(k = 1, 2, \dots)$ are real numbers**) be harmonic in the disk $|z| < R$. Then, by virtue of the author' s theorem (Theorem 2 from (?)), the function

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$$L^{(\omega_1,\dots,\omega_m)}[u(re^{i\varphi})] \equiv u_{(\omega_1,\dots,\omega_m)}(re^{i\varphi}) = \alpha_0 + \sum_{k=1}^{\infty}(\Delta_k^{(1)}\cdots\Delta_k^{(m)}\alpha_k\cos k\varphi -$$

$$-\Delta_k^{(1)}\cdots\Delta_k^{(m)}\beta_k\sin k\varphi)r^k$$

will be harmonic in the same disk $|z| < R$. Hence it follows that the function

$$u_{(\omega)}(re^{i\varphi};\tilde{\omega}) = u_{(\omega_1,\dots,\omega_m)}(re^{i\varphi};\tilde{\omega}_1,\dots,\tilde{\omega}_m) =$$

$$= \alpha_0 + \sum_{k=1}^{\infty}\left(\frac{\Delta_k^{(1)}\dots\Delta_k^{(m)}}{\widetilde{\Delta}_k^{(1)}\dots\widetilde{\Delta}_k^{(m)}}\alpha_k\cos k\varphi - \frac{\Delta_k^{(1)}\dots\Delta_k^{(m)}}{\widetilde{\Delta}_k^{(1)}\dots\widetilde{\Delta}_k^{(m)}}\beta_k\sin k\varphi\right)r^k$$

is also harmonic in the disk $|z| < R$.* Now Theorem 1 easily implies

Theorem 2. If the function

$$u(re^{i\varphi}) = \alpha_0 + \sum_{k=1}^{\infty}(\alpha_k\cos k\varphi - \beta_k\sin k\varphi)r^k$$

is harmonic in the disk $|z| < R$, then for every ρ $(0 < \rho < R)$ the integral formula

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} P_{(\tilde{\omega})}\left(\varphi-\theta,\frac{r}{\rho};\omega\right)u_{(\omega)}(\rho e^{i\theta};\tilde{\omega})\,d\theta$$

$$(0 \le r < \rho, \qquad 0 \le \varphi \le 2\pi).$$

is valid.

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named after N. K. Krupskaya

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* The Cauchy-Hadamard formula is used.
** They remain so in what follows as well.

CITED LITERATURE

[1] M. M. Dzhrbashyan, *Izv. AN SSSR, ser. matem.*, **32**, No. 5, 1075 (1968). [2] I. I. Bavrin, *DAN*, **187**, No. 3 (1969).

* The Cauchy–Hadamard formula is used.

Note: Figure translations are in progress. See original paper for figures.

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