THE BASIC EQUATION OF RADIATION ACOUSTICS AND THE SOLUTION OF THE CAUCHY PROBLEM

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Abstract

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HYDROMECHANICS

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THE BASIC EQUATION OF RADIATION ACOUSTICS AND THE SOLUTION OF THE CAUCHY PROBLEM

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1. The equations of hydromechanics in a weak radiation field interacting with a gas must be supplemented by including in the energy-transfer equation the specific heat influx q_r' due to the mechanism of radiative heat exchange (thermal emission and absorption).

In the case where Kirchhoff's law is applicable and scattering and the mechanical action of radiation are neglected,

$$q_r'(P,t) = \int_{(\nu)} k_\nu' \left\{ \int_{S^*} J_\nu'(P_s, t; -\mathbf{r}_S^0) e^{-r_{\tau S}} r_S^{-2} \cos(\mathbf{n}, \mathbf{r}_S^0)\, dS + \int_{V^*} k_\nu' B_\nu(P', t) e^{-r_\tau} r^{-2}\, dV - 4\pi B_\nu(P,t) \right\} d\nu, \tag{1}$$

$$r_\tau = \int_0^r k_\nu'\, dr, \qquad r_{\tau S} = \int_0^{r_S} k_\nu'\, dr,$$

where k_ν' is the volume absorption coefficient at frequency ν; B_ν is Planck' s function; (ν) is the frequency summation interval; S^* is the set of portions of boundary surfaces visible from the point $P(x_1, x_2, x_3)$; $\mathbf{n}$ is the outward normal to these surfaces; V^* is the set of all points P' visible from the point P; $\mathbf{r}, \mathbf{r}_S$ are the radius vectors PP' and PP_s; $J_\nu(P, t; \mathbf{r}_S^0)$ is the radiation intensity at frequency ν in the direction toward the point P from the visible point P_s of the boundary surface.

2. The propagation of small disturbances in a quiescent homogeneous equilibrium medium ($q_{r0}' = 0$, $J_{\nu 0}' = B_{\nu 0}$, $\mathbf{v}_0 = 0$) is described by the linearized equations of hydromechanics, which, for relative variations of the parameters $\rho'(P,t) = \rho_0[1 + \rho(P,t)]$, $v' = c_0\mathbf{v}$, etc., take the following form in dimensionless variables:

$$\nu\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{\gamma h_1}\nabla p-N_{\mathrm{Re}}^{-1}(\nabla\mathbf{v}-\nabla\operatorname{div}\mathbf{v})-\frac{X}{\nu}\nabla\operatorname{div}\mathbf{v}=0,$$

$$\nu\frac{\partial U}{\partial t}+\frac{c_0^2}{\gamma h_1}\operatorname{div}\mathbf{v}-h_4\frac{\theta}{\nu}\Delta T=\frac{1}{4}Zh_4R\{T\}, \tag{2}$$

$$\nu\partial\rho/\partial t+\operatorname{div}\mathbf{v}=0,\qquad p=h_1\rho+h_2T,\qquad U=h_3\rho+h_4T.$$

Here, in addition to generally accepted notation, the following designations are used:

$$R\{T\}=\int_{(\nu)}a_{\nu\tau_{\nu0}}\left[I_V(P,t)+I_S(P,t)-4T(P,t)\right]d\nu,$$

$$I_V=\int_{V^*}T(P',t)\frac{e^{-\tau_{\nu0}r}}{\pi r^2}\tau_{\nu0}\,dV, \tag{3}$$

$$I_S=\int_{S^*}b_\nu^{-1}J_\nu(P_s,t;-\mathbf{r}_S^0)\frac{e^{-\tau_{\nu0}r_S}}{\pi r_S^2}\cos(\mathbf{n},\mathbf{r}_S)\,dS;$$

$$h_1=\lambda_{pp},\quad h_2=\lambda_{pT},\quad h_3=\lambda_{Up},\quad h_4=c_{v3}T_0/U_0,\quad b_\nu=\lambda_{B_\nu T},$$

$$\lambda_{yx}=\left(\frac{\partial\ln y}{\partial\ln x}\right)_0,\quad a_\nu=\frac{1}{4}\frac{T_0}{B^*}\left(\frac{\partial B_\nu}{\partial T}\right)_0,\quad B^*=\int_{(\nu)}B_{\nu0}\,d\nu,$$

$$c_0^2=\gamma h_1p_0/\rho_0,\quad X=(\mu_1+2\mu)/\rho_0c_0^2t_0,\quad \theta=\lambda_0/c_{v0}c_0^2t_0,$$

$$Z=16\pi B^*/\rho_0c_{v0}T_0c_0,\quad N_{\mathrm{Re}}=\rho_0Lc_0/\mu_0. \tag{4}$$

The subscript 0 denotes parameters of the unperturbed medium; t_0 is the time scale.

All lengths and their differentials (including dV, dS) are dimensionless, referred to the corresponding power of the characteristic size L. The spectral frequency ν is left dimensional; the product $a_\nu d\nu$ is dimensionless. As L one may take the mean free path determining the radiation heat exchange, or the characteristic wavelength of the oscillatory motion, or the geometric size of the region. The function $J_\nu(P_s,t;-r_s^0)$ is determined by the optical properties of the boundary surface and the medium, by the boundary conditions, and by the geometry of the region under consideration. We shall regard it either as a known function of frequency, coordinates, and time, or as containing only linear operators acting on the temperature. In the case of black boundaries it is equal to $b_\nu T(P_s, t)$; in the case of "cold" boundaries it vanishes; for an unbounded medium $I_s = 0$. In the acoustic approximation, the motion of the boundary surfaces has no effect on the equations.

System (2) for the velocity potential reduces to a single fifth-order integro-differential partial differential equation—the fundamental equation of radiation acoustics

$$\left(\frac{\theta}{v^2}\Delta - \frac{\partial}{\partial t}\right) D(\varphi) + \frac{1}{4}\frac{Z}{v} R\{D(\varphi)\} = \frac{\gamma - 1}{\gamma}\frac{\partial}{\partial t}\Delta\varphi,$$

$$D(\varphi) \equiv \left(X\frac{\partial}{\partial t}\Delta + \frac{1}{\gamma}\Delta - v^2\frac{\partial^2}{\partial t^2}\right)\varphi. \tag{5}$$

All parameters are determined by the velocity potential:

$$\mathbf{v} = \nabla\varphi, \quad \frac{\partial p}{\partial t} = -\frac{\Delta\varphi}{v}, \quad \frac{\partial T}{\partial t} = \frac{\gamma h_1}{v h_2} D(\varphi). \tag{6}$$

System (2) can also be reduced to a single equation for determining the variation of the velocity, displacement, or temperature. In the last case

$$D\left\{\frac{\theta}{v^2}\Delta T - \frac{\partial T}{\partial t} + \frac{1}{4}\frac{Z}{v}R(T)\right\} - \frac{\gamma - 1}{\gamma}\frac{\partial}{\partial t}\Delta T = 0. \tag{7}$$

Equation (5) or (7) has the particular integral

$$\varphi(P, t) = \exp(\omega t)\psi(P), \tag{8}$$

where $\psi(P)$ satisfies equation (5), if in it $\partial/\partial t$ is replaced by ω and $\psi(P)$ is substituted for φ.

3. In space without boundary surfaces, equations (5), (7) possess bounded plane, spherical, and cylindrically symmetric integrals of the form (respectively in Cartesian, spherical, and cylindrical coordinate systems)

$$\varphi(x, t) = f(t)e^{i\lambda x}, \quad \varphi(r, t) = f(t)r^{-1}\sin(\lambda r),$$

$$\varphi(r, t) = f(t)J_0(\lambda r). \tag{9}$$

The function $f(t)$ must satisfy the equation

$$d_1 f \equiv \frac{d^3 f}{dt^3} + \frac{\lambda}{v}(X_1 + Q_1)\frac{d^2 f}{dt^2} + \frac{\lambda^2}{v^2}(1 + X_1 Q_1)\frac{df}{dt} + \frac{\lambda^3}{v^3}\frac{Q_1}{\gamma} f(t) = 0, \qquad (10)$$

$$X_1 = \frac{\lambda}{v} X, \qquad Q_1 = \frac{\lambda}{v}\theta + Z K(\lambda), \qquad K(\lambda) = \int_{(v)} a_v \frac{\tau_{v0}}{\lambda}\left(1 - \frac{\tau_{v0}}{\lambda}\,\mathrm{arctg}\,\frac{\lambda}{\tau_{v0}}\right) dv. \qquad (11)$$

The corresponding characteristic equation and the discriminant of its left-hand side ($f \sim \exp(\omega t)$, $\omega = \lambda n/v$) are represented by the expressions

$$n^3 + (X_1 + Q_1)n^2 + (1 + X_1 Q_1)n + Q_1/\gamma = 0, \qquad (12)$$

$$D = -108\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right), \qquad p = -\frac{1}{3}(X_1 + Q_1)^3 + 1 + X_1 Q_1;$$

$$q = \frac{2}{27}(X_1 + Q_1)^3 - \frac{1}{3}(X_1 + Q_1) + Q_1/\gamma. \qquad (13)$$

The particular Cauchy problem for equation (5), with initial conditions

$$t = 0, \qquad f(t) = f_0, \qquad f'(t) = f_1, \qquad f''(t) = f_2, \qquad f_k = \mathrm{const} \qquad (14)$$

and given λ, has solutions:

1) For $D > 0$, which can occur only in the case of not small X_1, greater than a certain limiting value,

$$f(t) = [(n_3 - n_2)(n_3 - n_1)(n_2 - n_1)]^{-1}\sum_{k=1}^{3} c_k e^{n_k t},$$

$$c_1 = (n_3 - n_2)[n_2 n_3 f_0 - (n_2 + n_3) f_1 + f_2],$$

$$c_2 = (n_1 - n_3)[n_3 n_1 f_0 - (n_3 + n_1) f_1 + f_2], \qquad (15)$$

$$c_3 = (n_2 - n_1)[n_1 n_2 f_0 - (n_1 + n_2) f_1 + f_2],$$

where n_k are the real roots of equation (12).

2) For $D = 0,\ q = 0$, which, for a given γ, can occur only at a single point (X_1, Q_1),

$$f(t) = \left[f_0 + (f_1 - f_0 n)t + \frac{1}{2}(f_2 - f_1 n)t^2\right] e^{nt}, \tag{16}$$

where n is a real triple root.

3) For $D = 0,\ q \neq 0$ (n_1 is simple, n_2 is a double real root),

$$f(t) = (n_2 - n_1)^{-1}\left[c_1 e^{n_1 t} + (c_2 + c_3 t)e^{n_2 t}\right],$$

$$c_1 = n_2^2 f_0 - 2n_2 f_1 + f_2,$$

$$c_2 = (n_1 - 2n_2)n_1 f_0 + 2n_2 f_1 - f_2, \tag{17}$$

$$c_3 = [n_1 n_2 f_0 - (n_1 + n_2)f_1 + f_2](n_2 - n_1).$$

4) For $D < 0$ (one real root, two complex conjugate roots), the solution is given by expression (15) as a superposition of two damped waves traveling in opposite directions and one aperiodic oscillation. In all cases $\mathrm{Real}\, n < 0$.

4. The Cauchy problem for equation (5) in the strip $H(|r| < \infty,\ 0 < t \leq t^0)$: find in the strip $H(|r| < \infty,\ 0 \leq t \leq t^0)$ a function satisfying equation (5) in H and the initial condition

$$t = 0, \qquad \varphi(P, t) = \varphi_c(P), \qquad \partial\varphi/\partial t = \varphi_1(P), \qquad \partial^2\varphi/\partial t^2 = \varphi_2(P), \tag{18}$$

where φ_n are given functions of the coordinates.

The Cauchy problem is posed in exactly the same way for equation (7). The functions φ_n determine the initial fields of velocities, pressures (accelerations), and derivatives $\partial T/\partial t$. In the analogous problem for displacements, the initial values determine the initial displacements, velocities, and accelerations of the particles.

Let us consider initial perturbations and solutions of the Cauchy problem with plane, spherical, or cylindrical symmetry in the corresponding coordinate systems (for the values k, respectively, $0, 1, 2$). By means of the Fourier transform in the plane case, the sine Fourier transform in the spherically symmetric case, and the Hankel transform in the cylindrically symmetric case, the basic equation and the initial conditions (18) are reduced to the Cauchy problem for an ordinary differential equation

$$d_1 f^{(k)} - v^{-2} F(t,\lambda) = 0,$$

$$t = 0, \qquad f^{(k)}(t,\lambda) = f_0^{(k)}(\lambda), \qquad df^{(k)}/dt = f_1^{(k)}(\lambda), \qquad d^2 f^{(k)}/dt^2 = f_2^{(k)}(\lambda); \tag{19}$$

$$f^{(0)}(t,\lambda) = \int_{-\infty}^{\infty} \varphi(x,t) e^{i\lambda x}\, dx, \qquad \varphi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^{(0)}(t,\lambda) e^{-i\lambda x}\, d\lambda,$$

$$f^{(1)}(t,\lambda) = \int_0^{\infty} \varphi(r,t) r \sin(\lambda r)\, dr, \qquad \varphi(r,t) = \frac{2}{\pi r}\int_0^{\infty} f^{(1)}(t,\lambda)\sin(\lambda r)\, d\lambda, \tag{20}$$

$$f^{(2)}(t,\lambda) = \int_0^{\infty} \varphi(r,t) r J_0(\lambda r)\, dr, \qquad \varphi(r,t) = \int_0^{\infty} f^{(2)}(t,\lambda)\lambda J_0(\lambda r)\, d\lambda;$$

$$F(t,\lambda) = \Lambda^{(k)}\left\{ \left[(X+\theta)\frac{\partial^2}{\partial t^2} + \left(1 + \frac{\lambda^2}{v^2} X\theta + \frac{\lambda}{v} XZK\right)\frac{\partial}{\partial t} + \frac{\lambda}{\gamma v}\left(\frac{\lambda}{v}\theta + ZK\right)\right]\varphi(r,t) - \frac{\theta}{v^2}\left(X\frac{\partial}{\partial t} + \frac{1}{\gamma}\right)\Delta\varphi(r,t)\right\},$$

$$\Lambda^{(0)}\varphi(x,t) = \left[e^{i\lambda x}(\partial/\partial x - i\lambda)\varphi(x,t)\right]_{x=-\infty}^{x=\infty}, \tag{21}$$

$$\Lambda^{(1)}\varphi(r,t) = \left[(\sin(\lambda r))\partial/\partial r - \lambda\cos(\lambda r)\right]\{r\varphi(r,t)\}\Big|_{r=0}^{r=\infty},$$

$$\Lambda^{(2)}\varphi(r,t) = \{r[J_0(\lambda r)\partial/\partial r + \lambda J_1(\lambda r)]\varphi(r,t)\}\Big|_{r=0}^{r=\infty}.$$

For $F(t,\lambda) = 0$, one obtains for the transforms the problem considered in (10), (14), where $f_n^{(k)}$ depend on λ. The domain (λ) of variation of λ $(0 < |\lambda| < \infty)$ is divided into four:

$$(\lambda) = (\lambda_1) + (\lambda_2) + (\lambda_3) + (\lambda_4) \tag{22}$$

corresponding to the four cases for the roots of equation (12).

For a nonviscous gas, when $\gamma < 9$, there remains a single domain (λ_4). The computation of the velocity-potential field has reduced to quadratures. The Cauchy problems for the temperature field or for other gas-dynamic functions are solved in exactly the same way.

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Note: Figure translations are in progress. See original paper for figures.