

# ON THE SMOOTHNESS OF WEAK SOLUTIONS FOR AN ELLIPTIC EQUATION DEGENERATING ON THE BOUNDARY

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE SMOOTHNESS OF WEAK SOLUTIONS FOR AN ELLIPTIC EQUATION DEGENERATING ON THE BOUNDARY

*(Presented by Academician S. L. Sobolev, May 28, 1969)*

In a domain  $G$  lying in the half-plane  $y > 0$ , consider the equation

$$Lu = yu_{yy} + u_{xx} + a(x, y)u_y + b(x, y)u_x + c(x, y)u = f(x, y), \quad (1)$$

whose boundary  $\Gamma$  consists of a smooth curve  $\sigma$ , resting on the axis  $y = 0$  by orthogonal segments at two points  $A, B$ , and the segment  $AB$  of the axis  $y = 0$ . Equation (1) is elliptic for  $y > 0$ , and on the axis  $y = 0$  it degenerates parabolically.

As is known <sup>(1,2)</sup>, the formulation of the first boundary value problem for equation (1) depends on the coefficient  $a(x, y)$ .

In the present paper we shall prove the existence and uniqueness of a weak solution from the spaces  $L_2$  and  $W_2^1$ , as well as the smoothness of these solutions under certain conditions on the coefficients of the equation and on the right-hand side. Let  $a(x, y), b(x, y) \in C^{1+\alpha}(\bar{G})$ ,  $c(x, y) \in C^\alpha(\bar{G})$ . By  $W_2^t, \dot{W}_2^t$  we denote the Sobolev spaces obtained by completing, respectively,  $C^\infty(\bar{G})$  and  $C_0^\infty(G)$  in the norm

$$\|u\|_t^2 = \int_G \sum_{|l| \leq t} (D^l u)^2 dG,$$

$$l = (l_1, l_2), \quad |l| = l_1 + l_2, \quad D^l u = \frac{\partial^{l_1+l_2}}{\partial^{l_1} x \partial^{l_2} y} u.$$

By  $C_L$  we denote the class of smooth functions in  $G$  satisfying the boundary conditions

$$u|_\Gamma = 0, \quad \text{if } a(x, 0) < 1;$$

$$u|_{\sigma} = 0, \quad \text{if } a(x, 0) \geq 1.$$

The closure of  $C_L$  in the norm  $W_2^1$  will be denoted by  $W_{2,L}^1$ , and the closure of  $L_2$  in the norm

$$\|f\|_{-1} = \sup_{v \in C_L} \frac{(f, v)_0}{\|v\|_{W_{2,L}^1}}$$

will be denoted by  $W_{2,L}^{-1}$ .

As is known <sup>(1,2)</sup>, the first boundary value problem for equation (1) is posed as follows:

$$Lu = f, \quad u|_{\Gamma} = 0, \quad \text{if } a(x, 0) < 1;$$

$$Lu = f, \quad u|_{\sigma} = 0, \quad \text{if } a(x, 0) \geq 1.$$

Let

$$L^*v = yv_{yy} + v_{xx} + (2 - a)v_y + bv_x + cv.$$

**Definition.** A function  $u \in L_2$  will be called a generalized solution of the first boundary value problem if the integral identity

$$(u, L^*v)_0 = (f, v)_0 \tag{2}$$

holds for any function  $v \in C_{L^*}$ .

Suppose that the conditions

$$2c - a_y - b_x < 0 \quad \text{and} \quad a(x, 0) < 0.5 \text{ or } a(x, 0) > 1.5 \tag{3}$$

are satisfied. Then in [7], under conditions (3), the inequality

$$\|L^*v\|_0 \geq c\|v\|_1, \quad c > 0 \text{ are constants, } v \in C_{L^*}, \tag{4}$$

was proved.

From [4] and inequality (4) it follows that

**Theorem 1.** *Suppose that conditions (3) are satisfied. Then for every  $f \in W_{2,L}^{-1}$ , there exists a unique generalized solution of the problem in  $L_2$ , and the following estimate for the solution holds*

$$\|u\|_0 \leq c\|f\|_{W_{2,L}^{-1}}.$$

**Theorem 2.** *Suppose that conditions (3) are satisfied. Then for every  $f \in L_2$  there exists a unique solution  $u \in W_{2,L}^1$ , and the estimate*

$$\|u\|_{W_{2,L}^1} \leq c\|f\|_0$$

*holds.*

We now show that, under certain conditions on the functions  $a, b, c, f$  and on the curve  $\sigma$ , which will be specified below, the generalized solution of the problem is classical.

Since for  $y > 0$  the equation is elliptic, we may apply to it the known results on the smoothness of generalized solutions of elliptic equations. Therefore the smoothness of the generalized solution must be investigated on the axis  $y = 0$ .

Take a rectangle  $K$  whose base coincides with the segment  $AB$  of the axis  $y = 0$ , and whose vertical sides belong to the segments by which the curve  $\sigma$  rests on the axis  $y = 0$ . By  $V(K)$  we denote the class of smooth functions equal to zero on the vertical sides, and also in a neighborhood of the upper side of the rectangle  $K$ .

It is easy to prove the following lemmas.

**Lemma 1.** *For functions of the class  $V(K)$  the inequality*

$$\|u\|_{t+1}^2 \leq c (\|Lu\|_{t-1}^2 + \|u_y\|_t^2 + \|u\|_0^2).$$

*holds.*

Hence we obtain

**Corollary 1.** *Let  $s = t + 1$ , and let a function  $v$ , in any domain  $G' \Subset K$ , belong to the class  $C^s(G')$  and vanish on the vertical sides and in a neighborhood of the upper side of the rectangle  $K$ . Suppose that  $v_y \in W_2^t(K)$ , and that the norm  $\|Lv\|_{t-1}$  is finite. Then  $v \in W_2^{t+1}(K)$ .*

**Lemma 2.** *There exists a rectangle  $K$  and a constant  $\lambda < 0$  such that for functions from  $V(K)$  the inequality*

$$(e^{\lambda y} Lv, v_y)_0 \geq c_1 \|v\|_1^2 - c_2 \|v\|_0^2, \quad c_1, c_2 > 0, \quad a(x, 0) > 1.5$$

*holds.*

**Corollary 2.** *Suppose that the conditions*

$$2c - a_y - b_x < 0, \quad a(x, 0) > 1.5 \quad (5)$$

are satisfied. Then for every function  $f \in L_2(K)$  and generalized solution  $u \in W_{2,L}^1$  the inequality

$$(e^{\lambda y} f, u_y)_0 \geq c_1 \|u\|_1^2 - c_2 \|u\|_0^2$$

holds.

**Theorem 3.** Let  $a, b, c, f \in C^{1+\alpha}(\bar{G})$  and suppose that conditions (5) are satisfied. Then a weak solution of the first boundary-value problem belongs to  $W_2^2(G) \cap C^0(\bar{G})$ .

**Proof.** It was proved above that, under conditions (5), the generalized solution  $u \in W_{2,L}^1$  is unique. Since for  $y > 0$  equation (1) is elliptic, in any domain  $G'$  obtained by intersecting the domain  $G$  with the half-plane  $y > \delta > 0$ , the solution will belong to  $C^{3+\alpha}(G')$ , provided that  $\sigma$  is a sufficiently smooth curve.

Take a function  $\xi(y) \in C^\infty(\bar{G})$  such that it differs from zero only in some rectangle  $P$  and, near the axis  $y = 0$ , is identically equal to one. The rectangle  $P$  belongs to the rectangle  $K$  indicated in Lemma 2.

Consider the function  $v = \xi u$ ; then  $v$  in a neighborhood of the upper side of the rectangle  $K$  is equal to zero and

$$Lv = \xi Lu + 2y\xi_{yy}u + uL\xi - c\xi u = F.$$

Since  $u$  is a smooth function inside the rectangle  $K$ , we have  $F \in W_2^1$ . Note that  $v_{yy} \in W_{2,L^*}^{-1}$ , and therefore there exists a sequence  $\{\varphi_n\}$  of functions  $\varphi_n \in L_2$  such that  $\varphi_n \rightarrow v_{yy}$  in the norm  $W_{2,L^*}^{-1}$ .

Consider the equation

$$Lv_n = \Phi - \varphi_n, \quad \Phi = F_y - a_{yy}y - b_{yy}x - c_{yy}. \quad (6)$$

Since  $\Phi - \varphi_n \in L_2$  and (5) is satisfied, there exists a unique solution of the first boundary-value problem in  $W_{2,L}^1$ . Note that  $\{v_n\}$  converges in  $L_2$  to  $v_y$ . Indeed, the right-hand side of equation (6) converges in  $W_{2,L^*}^{-1}$  to  $\Phi - v_{yy}$ . Hence  $\{v_n\}$  converges in  $L_2$  to some  $\varphi \in L_2$ , which is a generalized solution in  $L_2$  of (6). On the other hand,  $v_y$  satisfies equation (6) with right-hand side  $\Phi - v_{yy}$ , as well as the boundary conditions. By uniqueness of the solution it follows that  $\varphi = v_y$ .

From Corollary 2 we obtain the inequality

$$(e^{\lambda y} Lv_n, v_{ny})_0 \geq c_1 \|v_n\|_1^2 - c_2 \|v_n\|_0^2$$

or

$$(e^{\lambda y} \Phi, v_{ny})_0 \geq (e^{\lambda y} \varphi_n, v_{ny}) + c_1 \|v_n\|_1^2 - c_2 \|v_n\|_0^2$$

for any  $n$ . This means that there exists a constant  $c$  such that  $\|v_n\|_1 \leq c$ . Hence it follows that  $v_y \in W_2^1$ . Using Corollary 1, we conclude that  $v \in W_2^2(K)$ , i.e.,  $u \in W_2^2$ , and by the Sobolev embedding theorems  $u \in C^0(\bar{G})$ .

**Corollary 3.** Let  $a, b, c, f \in C^{l+\alpha}(\bar{G})$  and suppose that conditions (5) are satisfied. Then the generalized solution of the first boundary-value problem  $u \in W_2^1 \cap C^{l-1}(\bar{G})$ .

**Proof.** Consider in the rectangle  $K$  the equation

$$L_1 \varphi = y \varphi_{yy} + \varphi_{xx} + (a-1) \varphi_y + b \varphi_x + c \varphi = \Phi, \quad (7)$$

where  $\Phi$  is the function from Theorem 3. By Theorem 2, for equation (7) there exists a unique solution  $\varphi$  of the first boundary-value problem in  $W_{2,L}^1$ . On the other hand,  $v_y$  satisfies equation (7), the boundary conditions, and belongs to  $W_{2,L}^1$ , therefore  $\varphi = v_y$ . But equation (7) satisfies all the conditions of Theorem 3; hence  $v_y \in W_2^2$ , and by Corollary 1  $v \in W_2^3$ , and thus  $u \in W_2^3 \cap C^1(\bar{G})$ . Continuing in the same way, we obtain our assertion.

Consider, in the mixed domain  $\Omega = G + G_1$ , the equation

$$Lu = y u_{yy} + u_{xx} + a u_y + b u_x + c u = f, \quad (8)$$

where the boundary  $\gamma$  of the domain  $G_1$  consists of the segment  $AB$  of the axis  $y = 0$  and of two characteristics of equation (8),  $\Gamma_1$  and  $\Gamma_2$ , which emanate from one point  $C$  and are tangent to the axis  $y = 0$  at the points  $A$  and  $B$ , respectively.

On the basis of what has been said and of work (5), it is easy to prove the following theorem:

**Theorem 4.** Let  $a, b, c, f \in C^\infty(\Omega)$ ,  $2c - a_y - b_x < 0$ , and  $a(x, 0) > 1.5$  in  $G$ , and also let  $a, b, c$  in  $G_1$  be independent of  $y$ . Then the solution of equation (8), taking zero values on  $\sigma$ , is determined uniquely, is continuous in  $\bar{\Omega}$ , and in any  $\Omega' \subset \Omega$  one has  $u \in C^\infty(\Omega')$ .

In conclusion I consider it a pleasant duty to express my deep gratitude to S. A. Tersenov for the attention from which I benefited while writing the present work.

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*Note: Figure translations are in progress. See original paper for figures.*

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