A PROBABILISTIC CHARACTERIZATION OF ZERO-DIMENSIONAL COMPACT GROUPS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519

MATHEMATICS

V. M. MAKSIMOV

A PROBABILISTIC CHARACTERIZATION OF ZERO-DIMENSIONAL COMPACT GROUPS

(Presented by Academician A. N. Kolmogorov, May 7, 1970)

The basic convergence principle for compositions of non-identical distributions $\{\mu_m\}$ on a compact group G with a countable base asserts that the limiting points of the products $\mu_1 \dots \mu_n$, $n = 1, 2, \dots, \infty$, differ by right shifts by elements of G ([1]). In other words, there exist elements α_n of G such that the measures $\mu_1 \dots \mu_n \alpha_n$ converge as $n \to \infty$. This assertion can be strengthened somewhat by choosing α_n, with a certain margin, so that for any i the sequence $\mu_i \mu_{i+1} \dots \mu_n \alpha_n$ converges as $n \to \infty$. The element α_n depends not only on the measure μ_n, but also on the entire sequence $\{\mu_m\}$. The question is whether α_n can be chosen to depend only on the measure μ_n.

Definition. A set of measures A on a compact group G will be called **complete** if, for any distributions $x_i \in A$, $i = 1, 2, \dots, \infty$, (the measures x_i may coincide) the composition $x_1 x_2 \dots x_n$ converges as $n \to \infty$.

It follows from the definition that for any $x \in A$, the power x^n converges. Hence there follows the existence of classes A, for example consisting of the powers of a measure x for which x^n converges.

Now let μ be an arbitrary measure on G. Form the shifts of the measure $\{\mu c\}$, where c runs through all elements of G. We shall call this collection of shifts the class of shifts of the measure μ. Obviously, classes of shifts either do not intersect, or coincide.

Take one representative from each class of shifts and form from them the set A. Can the representatives be chosen so that the set A is complete? If this is possible on a certain group G, then the problem of convergence of compositions of non-identical measures on G is completely solved. Indeed, if from the class of shifts containing the measure μ one representative is chosen, denote it by $\mu c(\mu)$, where $c(\mu) \in G$ and is determined by the class of shifts containing the measure μ. Then for the sequence $\{\mu_n\}$ put $\alpha_1 = c(\mu_1)$, $\alpha_2 = c(\alpha_1^{-1} \mu_2)$, $\dots$, $\alpha_{n+1} = c(\alpha_n^{-1} \mu_{n+1})$. By virtue of the completeness of A, the product $\mu'_1 \dots \mu'_n = \mu_1 \dots \mu_n \alpha_n$, where $\mu'_i = \alpha_{i-1}^{-1} \mu_i \alpha_i$, converges as $n \to \infty$. That is, the completeness of A is a stronger property of measures on the group G than the convergence principle, and guarantees the existence of α_n depending only on the single measure μ_n.

Furthermore, the existence of the convergence principle for all measures on G is equivalent to the compactness of G ([1]). In this note we give a probabilistic characterization of zero-dimensional compact groups. Theorem 3 shows that if G is a positive-dimensional compact group, then it is impossible to make the set A complete under any choice of representatives from the classes of shifts. Conversely, if G is zero-dimensional, then this can already be done (Theorem 2).

Let B be an arbitrary complete set of measures. The following several assertions concerning B follow from the definition of completeness.

Lemma 1. If B is complete, then the semigroup of measures generated by B is also complete.

Lemma 2. If B is complete, then the closure of B in the weak topology of measures is also complete.

Lemma 3. *If B is complete, then $d^{-1}Bd$ is complete for any d in G.*

From Lemmas 1 and 2 it follows:

Theorem 1. *If a set of measures B on a compact group G is complete, then the smallest closed semigroup containing the set B is complete as well.*

Theorem 2. *If a compact group G is zero-dimensional, then one can compose a complete set of distributions A from representatives of each class of translates of measures on G.*

Proof. First let the group G be finite, consisting of s elements, and let μ be some measure on it. Then there is an element e whose probability in the measure μ is not less than $1/s$. From the class of translates containing the measure μ, take the distribution

$$\hat{\mu} = \mu e^{-1}.$$

The collection $\{\hat{\mu}\}$ over all possible classes forms a complete set. Indeed, for any sequence $\{\hat{\mu}_i\}$ from this set, the probability of the identity G in each measure is not less than $1/s$. Therefore, by virtue of ([2]), the product

$$\hat{\mu}_1 \dots \hat{\mu}_n$$

converges as $n \to \infty$, i.e., the theorem is proved for finite groups.

Now let G be an arbitrary zero-dimensional compact group with an infinite number of elements. Then, according to ([3]), the group G is represented by a convergent sequence of finite groups

$$G_1,\ G_2, \dots,\ G_i, \dots; \quad i = 1, 2, \dots, \infty,$$

i.e., homomorphisms φ_i of the group G_{i+1} onto G_i are given, and the group G then proves to be isomorphic to G_ω, the group of sequences

$$(x_1, x_2, \dots, x_n)$$

with componentwise multiplication, where $x_i \in G_i$ and

$$x_i = \varphi_i \varphi_{i+1} \dots \varphi_{j-1}(x_j).$$

A base of neighborhoods of G_ω consists of sets of the form $[U_\alpha]$, where U_α is a neighborhood in G_α, and $[U_\alpha]$ is the collection of sequences in which the coordinates with number α belong to U_α. Denote the number of elements of G_i by

$$s_i = |G_i|.$$

Then $s_i \leq s_{i+1}$. Since G is infinite, G_ω is also infinite, and therefore $s_i \to \infty$.

By virtue of the isomorphism of G_ω and G, it is enough to prove Theorem 2 for the group G_ω. Let μ be a measure on G_ω. If $b_\alpha \in G_\alpha$, then the measure

$$\mu_\alpha(b_\alpha) = \mu[b_\alpha]$$

defines a measure on G_α and is called the projection of μ onto G_α.

Define an element

$$b = (b_1, b_2, \dots)$$

of G_ω so that the probability of $b_\alpha \in G_\alpha$ in the projection μ_α of the measure μ would be not less than $1/s_\alpha$. For b_1 take the element of G_1 having the largest probability in μ_1. Since $|G_1| = s_1$, we have

$$\mu_1\{b_1\} \geq 1/s_1.$$

Suppose that the first k elements

$$b_1, b_2, \dots, b_k$$

have already been defined, for which

$$\mu_i(b_i) \geq 1/s_i, \qquad \varphi_i(b_{i+1}) = b_i.$$

Define the element b_{k+1}. To this end, consider the complete inverse image of the element b_k under the homomorphism φ_k. Denote it by B_k. B_k contains

$$s_{k+1}/s_k = r_k$$

elements. By the construction of the projections μ we have

$$\mu_{k+1}(\{B_k\}) = \mu_k(b_k) \geq 1/s_k.$$

Consequently, in B_k there is an element b_{k+1} for which

$$\mu_{k+1}(b_{k+1}) \geq 1/s_k r_k = 1/s_{k+1}.$$

Thus, the required element

$$b = (b_1, b_2, ...)$$

exists.

Denote

$$b^{-1} = (b_1^{-1}, b_2^{-1}, ...)$$

by $c(\mu)$, and from the class of translates containing the measure μ, choose the representative equal to

$$\mu c(\mu).$$

Now take one arbitrary measure μ from each class of translates and form A, consisting of the collection of measures

$$\{\mu c(\mu)\}.$$

To prove Theorem 2 it remains to show that the set of measures A is complete.

Let

$$\{\mu_m c(\mu_m)\}$$

be an arbitrary sequence of measures from A. In order that

$$\prod \mu_m c(\mu_m)$$

converge, it is enough that the products of the Fourier coefficients of these measures converge for any finite-dimensional representations φ of the group G. But this convergence will follow from the convergence of the product of measures

$$\prod \mu_m(g),$$

where $\mu_m(g)$ are measures on the group

$$g = \varphi(G_\omega),$$

induced by the measures $\mu_m c(\mu_m)$ under the mapping φ.

Let N be the kernel of the representation φ. The factor group

$$g = G/N$$

is finite, for otherwise the group G_ω would not be zero-dimensional. Therefore, if the probabilities of the identity in the measures $\mu_m c(\mu_m)$ are uniformly bounded below by some positive number ε, then, by virtue of (^{2}), the product of measures

$$\prod \mu_m(g)$$

converges. But the probability of the identity in the measure $\mu_m(g)$ is obviously equal to the probability of the subgroup N in the measure $\mu_m c(\mu_m)$.

It can be shown that the group N contains the set $[e_n]$ for some n, where e_n is the identity of G_n. The probability of $[e_n]$ in the measure $\mu_m c(\mu_m)$, by the definition of the projection of measures, is equal to the probability of the identity in the projection $\mu_m c(\mu_m)$ onto G_n. But this probability, in view of $c(\mu) = (b_1^{-1}, b_2^{-1}, ...)$, is equal to the probability of the set $[b_n]$ in μ_m, which by construction is not less than $1/s_n$. But $[e_n] \subseteq N$. Therefore the probability of the subgroup N in $\mu_m c(\mu_m)$ is not less than $1/s_n$ for all $m = 1, 2, \dots, \infty$. The theorem is proved.

Theorem 3. *For distributions on a compact group G of positive dimension, it is impossible to form a complete set of distributions from representatives of each class of shifts.*

We shall first prove two auxiliary propositions.

Lemma 4. *There exists a sequence of measures on the circle $\mu_1, \mu_2, ...$ such that no representatives of the shift classes containing the measures $\mu_1, \mu_2, ...$ form a complete set.*

Proof. Let μ_n be a distribution on the circle in which, with probability 1/2, the element $\exp\{i2\pi/f(n)\}$ is taken, and with probability 1/2 the identity element of the group is taken. If $f(n)$ satisfies the conditions: $f(n)$ are even positive integers, $\sum 1/f(n) = \infty$, $\sum 1/f(n)^2 < \infty$, then the measures μ_n satisfy the condition of the lemma. The proof of this is based on the fact that a representative of the shift class containing the measure μ_n must be of the form $\exp\{i\frac{2\pi}{f(n)}m(n)\}\mu_n$, where $m(n)$ is a positive integer not exceeding $f(n)$. Otherwise the degree of this representative would not converge, which would violate the completeness of the set of representatives.

Lemma 5. *Let g be a commutative subgroup of G, and let A be a complete set of measures consisting of representatives of all shift classes. Then for any countable sequence of measures $\mu_1, \mu_2, ...$ with supports in g, there exist elements $a_1, a_2, ...$ belonging to g, and an element d of G, such that the measures $d^{-1}(\mu_1 a_1)d, d^{-1}(\mu_2 a_2)d, ...$ belong to A.*

Proof. For simplicity, we first consider the case of two measures μ_1 and μ_2. From these measures construct the sequence of distributions $\mu_1 b_1, b_1^{-1}\mu_2 b_2, \dots, b_{2n-1}^{-1}\mu_2 b_{2n}, ...$, where b_n and b_{n+1} are connected by the relations $b_{2n} = c(b_{2n-1}^{-1}\mu_2)$ and $b_{2n-1} = c(b_{2(n-1)}^{-1}\mu_1)$.

The function $c(\cdot)$ is defined on G so that the measure $\mu c(\mu)$ is a representative of the shift class containing the measure μ. Therefore the products of the first $2n-1$ and $2n$ terms of this sequence, equal respectively to $(\mu_1\mu_2)^{n-1}\mu_1 b_{2n-1}$ and $(\mu_1\mu_2)^n b_{2n}$, have the same limits as $n \to \infty$.

By the convergence principle, the powers $(\mu_1\mu_2)^n$ will converge to the invariant measure on the group g_1, $g_1 \subseteq g$, if they are shifted by the corresponding elements α_n, $\alpha_n \in g$. Since $(\mu_1\mu_2)^n \alpha_n \alpha_n^{-1} b_{2n}$ converges, it follows that $\alpha_n^{-1} b_{2n}$ must be equal to $g_1^{(n)} d_n^{(1)}$, where $g_1^{(n)} \subseteq g_1 \subseteq g$, and the element $d_n^{(1)}$ tends to some $d \in G$. Similarly, $(\mu_1\mu_2)^{n-1}\mu_1 b_{2n-1} = \mu_1(\mu_1\mu_2)^{n-1}\alpha_{n-1}\alpha_{n-1}^{-1}b_{2n-1}$ (by the commutativity of g) converges to the same limit as $(\mu_1\mu_2)^n b_{2n}$. Since the support of μ_1 is contained in g, and g is commutative, we obtain $b_{2n-1} = g_2^{(n)} d_n^{(2)}$, where $g_2^{(n)} \subseteq g$, and $d_n^{(2)}$ tends to $d \in G$.

Taking now into account the form of the elements b_{2n-1} and b_{2n} and the compactness of the group G, one can find indices n_i such that $g_1^{(n_i)} \to q_1$, $g_2^{(n_i)} \to q_2$, $g_1^{(n_i-1)} \to q_1'$, $g_2^{(n_i-1)} \to q_2'$. It is clear that all these limits belong to g. Then $b_{2n_i}^{-1}\mu_2 b_{2n_i} \to d^{-1}q_1^{-1}\mu_2 q_1 d$, which, by the commutativity of g, can be written in the form $d^{-1}\mu_2(q_1 q_2^{-1})d$. Similarly, $b_{2(n_i-1)}^{-1}\mu_1 b_{2n_i-1} \to d^{-1}(q_2')^{-1}\mu_1(q_1')d = d^{-1}\mu_1(q_2')^{-1}q_1'd$. Denote $q_1 q_2^{-1} = a_1 \in g$, $q_1'(q_2')^{-1} = a_2 \in g$. By Theorem 1,

the limit measures $d^{-1}(\mu_1\alpha_1)d$ and $d^{-1}(\mu_2\alpha_2)d$ belong to A. Thus Lemma 5 is proved for two arbitrary measures μ_1 and μ_2.

The case of a larger finite number of measures $\mu_1, \mu_2, \dots, \mu_n$ with supports belonging to g is, obviously, considered analogously.

Thus, for any initial measures $\mu_1, \mu_2, \dots, \mu_n$ of the sequence $\{\mu_m\}$, there will be found elements $\alpha_1^{(n)}, \dots, \alpha_n^{(n)} \in g$ and $d_n \in G$ such that the measures

$$d_n^{-1}(\mu_i \alpha_i^{(n)}) d_n$$

belong to A for $i = 1, 2, \dots, n$.

Let now, for the sequence $\{d_n\}$, the element d of G be a limit point. Then for the pair $\{d_n, \alpha^{(n)}\}$ there exists a limit point $\{d, \alpha_i\}$, where $\alpha_i \in g$.

Since the measure $d_n^{-1}(\mu_i\alpha_i^{(n)})d_n \in A$, by virtue of Theorem 1 the measure $d^{-1}(\mu_i\alpha_i)d$ also belongs to A. The lemma is proved.

We now pass directly to the proof of Theorem 3.

Let the group G be of nonzero dimension. Then there exists a linear representation φ of the group G with kernel N such that G/N is a Lie group.

Let $\varphi(\mu)$ be the measure on G/N generated by the measure μ under the mapping φ. Suppose that Theorem 3 is not fulfilled for the group G, i.e., there exists a complete set A of representatives of each class of shifts on G. But from the completeness of A follows the completeness of $\varphi(A)$, which also consists of representatives of all classes, but now on the group G/N. Thus there exists a Lie group $G_0 = G/N$ for which Theorem 3 does not hold. In every Lie group there is a subgroup that is a circle. Denote one of such subgroups G_0 by g. On g consider measures $\mu_1, \mu_2, \dots$ satisfying the conditions of Lemma 4. According to Lemma 5, there exist elements $\alpha_1, \alpha_2, \dots \in g$ and $d \in G_0$ such that the measures $d^{-1}(\mu_i\alpha_i)d$, $i = 1, 2, \dots, \infty$, belong to A. The collection of measures $d^{-1}(\mu_i\alpha_i)d$, $i = 1, 2, \dots, \infty$, must be complete as a subset of the complete set A. But then, by virtue of Lemma 3, the sequence $\{\mu_i \alpha_i\}$ is complete on the circle g, which contradicts Lemma 4. The theorem is proved.

Institute of Chemical Physics
Academy of Sciences of the USSR
Moscow

Received
2 IV 1970

REFERENCES

1. V. V. Sazonov, V. N. Tutubalin, *Theory of Probability and Its Applications*, **11**, no. 1 (1966).

2. V. M. Maksimov, *Theory of Probability and Its Applications*, **13**, no. 2 (1968).

3. L. S. Pontryagin, *Continuous Groups*, Moscow, 1954.

Note: Figure translations are in progress. See original paper for figures.