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Abstract

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A NONLINEAR ANALOGUE OF THE NEWTONIAN POTENTIAL AND METRIC PROPERTIES OF (p, l)-CAPACITY

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In many questions of analysis, the massiveness of a set E located in Euclidean space R^n is naturally characterized by means of the quantity

$$\inf\{\|\nabla_l \varphi\|_p^p : \varphi \in \mathcal{U}(E)\},$$

where $\| \ \|_p$ is the norm in $L_p = L_p(R^n)$, $\mathcal{U}(E) = \{\varphi \in \mathcal{D} : \varphi(x) \geq 1$ for any $x \in E\}$, $\mathcal{D}$ is the set of all functions of class $C^\infty(R^n)$ with compact supports, and $\nabla_l \varphi$ is the gradient of order l of the function φ (see, for example, ($^{1-14}$)). This quantity, called the (p, l)-capacity of the set E and denoted below by $\operatorname{cap}_{p,l}(E)$, for $p = 2$, $l = 1$ and $n \geq 3$ coincides with the classical capacity studied in potential theory (see ($^{15-17}$)).

In this paper an extension of the Newtonian potential is proposed which, in our opinion, is well adapted to the study of the function $\operatorname{cap}_{p,l}$. With the help of this extension it has been possible to find metric conditions necessary and (separately) sufficient for the vanishing of (p, l)-capacity (see below, item 5). For $p = 2$ these conditions coincide with the known conditions of the potential theory of M. Riesz (see (^{15}), Ch. III, Theorems 3.13 and 3.14; (^{16}), § IV). The theorems of item 5 refine some results of the papers (^{6,7,14}).

Throughout what follows we shall assume that $p > 1$, l is a positive (not necessarily integral) number, and that $pl < n$.

1. The function $c_{p,l}$ and the space $\mathcal{L}_p^l$. For a function φ of class $\mathcal{D}$ put

$$\Lambda^l_{x\to y}\varphi = F^{-1}_{\xi\to y}(|\xi|^l F_{x\to\xi}(\varphi)),$$

where F is the Fourier transform. We introduce in $\mathcal{D}$ the norm

$$\|\varphi\|_{p,l} = \|\Lambda^l \varphi\|_p.$$

We define on the system of all compact subsets of the space R^n the capacity $c_{p,l}$:

$$K \xrightarrow{c_{p,l}} \inf\{\|\varphi\|_{p,l}^p : \varphi \in \mathcal{U}(K)\}.$$

Let us note that the ratio of $c_{p,l}(K)$ to $\operatorname{cap}_{p,l}(K)$ is bounded above and separated from zero uniformly with respect to all compact sets $K \subset R^n$, since for integral l the norm $\| \ \|_{p,l}$ on $\mathcal{D}$ is equivalent to the norm $\varphi \mapsto \|\nabla_l \varphi\|_p$. Starting from the set function $c_{p,l}$, we define in the usual way the inner and outer capacities $\underline{c}_{p,l}(E)$ and $\overline{c}_{p,l}(E)$ for any subset E of the space R^n (see ($^{15-18}$)). If E is an analytic (and, in particular, Borel) set, then

$$\overline{c}_{p,l}(E) = \underline{c}_{p,l}(E).$$

In this case, instead of $\overline{c}_{p,l}(E)$ we shall write $c_{p,l}(E)$.

We denote by $\mathcal{L}_p^l$ the completion of the space $\mathcal{D}$ with respect to the norm $\| \ \|_{p,l}$. This space can be embedded naturally in L_s, where

$$s = np(n - pl)^{-1}.$$

We shall assume that the operator $\Lambda^l : \mathcal{D} \to L_p$ has been extended by continuity to the whole space $\mathcal{L}_p^l$.

Lemma 1. *The operator Λ^l maps $\mathcal{L}_p^l$ isometrically onto L_p. The operator inverse to Λ^l (denoted by Λ^{-l}) is given by the equality*

$$(\Lambda^{-l} f)(x) = c \int \frac{f(y)}{|x-y|^{n-l}}\, dy \quad (x \in R^n)$$

(c is a constant depending only on n, p, l; the integral is taken over R^n).

2. Energy

It is easy to see that every functional $T \in (\mathcal{L}_p^l)^*$ can be identified with some generalized function in the sense of Schwartz. The energy of a generalized function $T \in (\mathcal{L}_p^l)^*$ will mean the number

$$\mathcal{E}_{p,l}(T) \overset{\text{def}}{=} \|T\|^q_{(\mathcal{L}_p^l)^*}, \quad \text{where } q = p(p-1)^{-1}.$$

By the word measure we shall everywhere below mean a nonnegative countably additive locally finite function, defined on the Borel σ-ring of the space R^n. Identifying the functional $\varphi \to \int \varphi \, d\mu$ ($\varphi \in \mathcal{D}$) with the measure μ, we can formulate the following result.

Lemma 2. A measure μ is a generalized function with finite (p,l)-energy if and only if

$$\int \left[\int \frac{d\mu(y)}{|x-y|^{n-l}} \right]^q dx < +\infty \quad \left(q = \frac{p}{p-1} \right).$$

The set of all such measures (denoted below by $\mathfrak{M}_{p,l}$) is closed in the space $(\mathcal{L}_p^l)^*$.

The last assertion, for $p = 2$ and $l = 1$ ($n \geqslant 3$), turns into the well-known theorem of A. Cartan (see (^{15}), p. 117, Theorem 1.18).

3. Potentials

Let $T \in (\mathcal{L}_p^l)^*$. Put

$$U_{p,l}^T = \Lambda^{-l}\left(\left|(\Lambda^{-l})^* T\right|^{(2-p)/(p-1)} (\Lambda^{-l})^* T\right)$$

(the asterisk denotes passage to the adjoint operator). It is easy to see that $U_{p,l}^T \in \mathcal{L}_p^l$. The function $U_{p,l}^T$ will be called the (p, l)-potential of the generalized function T. Note that for $p = 2$ we return to the well-known definition of the Riesz potential of a generalized function (^{15}), p. 434). If $\mu \in \mathfrak{M}_{p,l}$, then for almost all $x \in R^n$

$$U_{p,l}^\mu(x) = c \int \left[\int \frac{d\mu(z)}{|z-y|^{n-l}}\right]^{1/(p-1)} \frac{dy}{|y-x|^{n-l}}.$$

Let us also record the expression for the energy of a measure $\mu \in \mathfrak{M}_{p,l}$:

$$\mathcal{E}_{p,l}(\mu) = \int U_{p,l}^\mu \, d\mu.$$

Lemma 3. The image of the space $(\mathcal{L}_p^l)^*$ under the mapping $T \to U_{p,l}^T$ is equal to $\mathcal{L}_p^l$.

Theorem 1 (rough maximum principle). Let the measure μ be such that $U_{p,l}^\mu(x) \leq M$ for every x in the (closed) support of the measure μ. Then for every $x \in R^n$ the inequality $U_{p,l}^\mu(x) \leq cM$ holds, where c depends only on n, p, and l.

Theorem 2. Let μ be a measure with compact support. If $U_{p,l}^\mu(x) < +\infty$ for μ-almost all $x \in R^n$, then for every $\varepsilon > 0$ one can indicate such a compact set $K \subset R^n$ that the (p, l)-potential of the restriction of the measure μ to the set K is continuous in R^n and $\mu(R^n \setminus K) < \varepsilon$.

4. Capacitary distribution

Let K be a compact set in R^n, and let $\{\varphi_m\}$ be some sequence minimizing the norm of the space $\mathcal{L}_p^l$ on the set $\mathcal{U}(K)$:

$$\lim \|\varphi_m\|_{p,l}^p = C_{p,l}(K), \quad \varphi_m \in \mathcal{U}(K), \quad \text{for all } m.$$

From the uniform convexity of the space $\mathcal{L}_p^l$ it follows that this sequence converges in $\mathcal{L}_p^l$ to a certain function $V_K \in \mathcal{L}_p^l$, which does not depend on the choice of the sequence $\{\varphi_m\}$.

Theorem 3. The function V_K is the (p,l)-potential of some measure $\mu_K \in \mathfrak{M}_{p,l}$, concentrated on K, and such that:

1) $U_{p,l}^{\mu_K}(x) \geqslant 1$ for every $x \in K$, except for a set of points of zero capacity $c_{p,l}$;
2) $U_{p,l}^{\mu_K}(x) \leqslant 1$ for every $x \in \operatorname{supp} \mu_K$;
3) $\mu_K(K) = \int U_{p,l}^{\mu_K}\, d\mu_K = c_{p,l}(K)$.

From this theorem it is not hard to derive that the measure μ_K and the capacity $c_{p,l}(K)$ are solutions of certain extremal problems analogous to the classical ones ([15]), definition on p. 169, the definition of Vallée-Poussin on p. 176). It can be shown that, analogously to the classical theory of potential, for every bounded subset E of the space R^n there exist "outer" and "inner" capacitary distributions, which coincide with one another if E is an analytic set (see ([15]), Ch. II, Theorems 2.6 and 2.7).

5. Metric characteristics of sets of zero (p,l)-capacity. Denote by $m_h(E)$ the Hausdorff h-measure of the set E (see ([15]), pp. 244-245). If $h(t) = t^\alpha$, then instead of $m_h(E)$ we shall write $m_\alpha(E)$.

Theorem 4. Suppose that

$$\int_0^1 \left(\frac{h(s)}{s^{n-lp}}\right)^{1/(p-1)} \frac{ds}{s} < +\infty.$$

If E is a Borel set $(E \subset R^n)$ for which $m_h(E) > 0$, then $c_{p,l}(E) > 0$.

For the proof of this theorem we verify that the potential of a measure concentrated on E is bounded (cf. ([16]), p. 29).

Theorem 5. If $m_{n-lp}(E) < +\infty$ (E is a Borel set in R^n), then $c_{p,l}(E) = 0$.

Consider now the n-dimensional Cantor set e, equal to the intersection of a decreasing sequence of sets $\{e_k\}_{k=1}^\infty$, where e_k is equal to the sum of 2^{kn} closed cubes with side l_k.

Theorem 6. The condition

$$\sum_{k=1}^\infty 2^{-kn/(p-1)} l_k^{-(n-pl)/(p-1)} = +\infty$$

is necessary and sufficient in order that the set e have zero capacity $c_{p,l}$.

This theorem generalizes Otsuka's theorem (see ([13]), p. 31). In the proof of Theorem 7 the following lemma is used, which generalizes a theorem of Nevanlinna (see ([16]), p. 30) and is based on an estimate of the (p, l)-potentials of certain measures.

Lemma 4. If for every $r > 0$ the Borel set $E \subset R^n$ can be covered by $A(r)$ closed balls of radii not exceeding r, and if

$$\int_0 [A(r) r^{n-lp}]^{-1/(p-1)} \frac{dr}{r} = +\infty,$$

then $c_{p,l}(E) = 0$.

6. Removal of singularities of analytic functions. Let D be a domain on the extended complex plane, containing ∞, and let $H_p(D)$ be the set of all functions holomorphic in D and belonging to $L_p(D)$. It is not difficult to verify that $H_p(D)$ consists only of the zero function if and only if $c_{q,1}(E) = 0$. Therefore the following assertion is valid, supplementing a theorem of Carleson ([16]), p. 73):

Theorem 7. Let $p > 2$. The set $H_p(D)$ contains only the zero function if there exists a function h such that $m_h(R^2 \setminus D) > 0$, and

$$\int_0^1 \left[\frac{h(t)}{t}\right]^{p-1} dt < +\infty.$$

In a similar way one can sharpen certain known theorems on the removability of singularities of solutions of elliptic equations, formulated in terms of Hausdorff measures (see ([5,7,16])).

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Note: Figure translations are in progress. See original paper for figures.

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