ON A PROPERTY OF FUNCTIONS OF TWO VARIABLES AND MULTIPLE FOURIER SERIES

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Abstract

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ON A PROPERTY OF FUNCTIONS OF TWO VARIABLES AND MULTIPLE FOURIER SERIES

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In the present article one remarkable feature of functions of two variables and of trigonometric double Fourier series is established. In particular, a continuous function $F(x,y)$ of two variables is constructed, having, at almost every point, an arbitrarily prescribed "rate of change" with respect to the aggregate of the variables (x,y) and with respect to each variable separately. Further, the differentiated Fourier series of the indicated function, depending on the method of differentiation, sums almost everywhere to different arbitrarily fixed and mutually independent measurable functions.

For the statement of the results obtained, the following notation is adopted in this work: $R_0 = [0 \le x \le 1;\ 0 \le y \le 1]$; $R = [-\pi \le x \le \pi;\ -\pi \le y \le \pi]$; $C(P;r)$ is the circle of radius r with center at the point $P(x,y)$; $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$;

$$\Delta(F;\ x,y,h,l) = \frac{[F(x+h,y+l) - F(x,y+l) - F(x+h,y) + F(x,y)]}{hl},$$

$$\widetilde{\Delta}(F;\ x,y,h,l) = \frac{[F(x+h,y+l) + F(x-h,y+l) + F(x+h,y-l) + F(x-h,y-l) - 4F(x,y)]}{h^2+l^2};$$

$$C_1\Delta(F;\ x,y,h,l) = \frac{4}{h^2 l^2}\int_0^h \int_0^l [F(t,\tau) - F(x,\tau) - F(t,y) + F(x,y)]\,dt\,d\tau,$$

$$C_1\widetilde{\Delta}(F;\ x,y,h,l) = \frac{3}{hl(h^2+l^2)}\int_0^h \int_0^l [F(x+t,y+\tau) + F(x-t,y+\tau)+$$
$$+F(x+t,y-\tau) + F(x-t,y-\tau) - 4F(x,y)]\,dt\,d\tau,$$

$$\Delta_r^*(F;\ x,y) = \frac{2}{\pi r^3}\int_{C(P;r)} [F(t,\tau) - F(x,y)]\,dS,$$

$$\Delta_{21}(F;\ x,y,h,l) = \frac{1}{2h^2 l}\int_0^h \int_0^l [F(x+t,y+\tau) - F(x-t,y+\tau)+$$

$$+F(x+t,y-\tau)-F(x-t,y-\tau)]\,dt\,d\tau,$$

$$\Delta_{12}(F;\ x,y,h,l)=\frac{1}{2hl^2}\int_0^h\int_0^l[F(x+t,y+\tau)-F(x+t,y-\tau)+$$

$$+F(x-t,y+\tau)-F(x-t,y-\tau)]\,dt\,d\tau,$$

$$\Delta_r^{(2)}(F;\ P)=\frac{1}{4\pi^2r^2}\int_{C(P;r)}dS_Q\int_{C(Q;r)}F(M)\,dS_M-\frac{1}{\pi r}\int_{C(P;r)}F(Q)\,dS_Q+F(P).$$

We shall say that $\{h_n\}$ and $\{l_m\}$ are λ-sequences if, from the relation $1/\lambda\le m/n\le\lambda$, it follows that

$$0<A(\lambda)\le|h_n/l_m|\le B(\lambda)<\infty.$$

The symbol $(h,l)_\lambda\to 0$ means that $h\to 0,\ l\to 0$ and $1/\lambda\le|h/l|\le\lambda,\ \lambda\ge 1$.

Next, we shall consider the following derivatives of a function of two variables:

$$D_\lambda F(x,y)=\lim_{(h,l)_\lambda\to 0}\Delta(F;x,y,h,l),\qquad \widetilde{D}F(x,y)=\lim_{h,l\to 0}\widetilde{\Delta}(F;x,y,h,l),$$

$$\widetilde{D}_\lambda F(x,y)=\lim_{(h,l)_\lambda\to 0}\widetilde{\Delta}(F;x,y,h,l),\qquad C_{1\lambda}DF(x,y)=\lim_{(h,l)_\lambda\to 0}C_1\Delta(F;x,y,h,l),$$

$$C_{1\lambda}\widetilde{D}F(x,y)=\lim_{(h,l)_\lambda\to 0}C_1\widetilde{\Delta}(F;x,y,h,l),\qquad \Delta^*F(x,y)=\lim_{r\to 0}\Delta_r^*(F;x,y),$$

$$D_{21}^\lambda F(x,y)=\lim_{(h,l)_\lambda\to 0}\Delta_{21}(F;x,y,h,l),$$

$$D_{12}^\lambda F(x,y)=\lim_{(h,l)_\lambda\to 0}\Delta_{12}(F;x,y,h,l),$$

$$\Delta^{*(2)}F(x,y)=\lim_{r\to 0}\frac{16\Delta_r^{(2)}(F;x,y)}{r^4}.$$

Theorem 1. Let $f_i(x,y)$, $i=1,2,3,4$, be arbitrary measurable and almost everywhere finite functions on R_0. Then there exists a continuous function $F(x,y)$ such that almost everywhere on R_0:

$$D_{21}^\lambda F(x,y)=f_1(x,y),\qquad D_{12}^\lambda F(x,y)=f_2(x,y),$$

$$D_\lambda F(x,y) = f_3(x,y), \qquad \widetilde{D}_\lambda F(x,y) = f_4(x,y).$$

We note that in Theorem 1 the derivative $D_\lambda F(x,y)$ may be replaced by the derivative $C_{1\lambda} DF(x,y)$, and $\widetilde{D}_\lambda F(x,y)$ by the derivative $C_{1\lambda}\widetilde{D}F(x,y)$ or $\Delta^* F(x,y)$.

Theorem 2. Let $\varphi(x,y)$ be an arbitrary measurable and almost everywhere finite function on R_0. Then there exists a continuous function $\Phi(x,y)$ such that almost everywhere on R_0

$$\widetilde{D}\Phi(x,y) = \varphi(x,y).$$

Theorem 3. Let $\{h_n\}$ and $\{l_m\}$, $h_n \neq 0$, $l_m \neq 0$, be arbitrary λ-sequences tending to zero. There exists a continuous function $F(x,y)$ such that, whatever measurable and almost everywhere finite on R_0 functions $f_i(x,y)$, $i = 1,2,3,4$, may be, there exist λ-subsequences $\{h_{n_k}\}$ and $\{l_{m_j}\}$ for which, almost everywhere on R_0, the relations

$$\lim_{(n_k,m_j)_\lambda \to \infty} \Delta(F; x, y, h_{n_k}, l_{m_j}) = f_1(x,y),$$

$$\lim_{(n_k,m_j)_\lambda \to \infty} \widetilde{\Delta}(F; x, y, h_{n_k}, l_{m_j}) = f_2(x,y),$$

$$\lim_{(n_k,m_j)_\lambda \to \infty} \Delta_{21}(F; x, y, h_{n_k}, l_{m_j}) = f_3(x,y),$$

$$\lim_{(n_k,m_j)_\lambda \to \infty} \Delta_{12}(F; x, y, h_{n_k}, l_{m_j}) = f_4(x,y).$$

We note that in Theorem 3, $\Delta(F; x, y, h, l)$ may be replaced by the expression $C_1\Delta(F; x, y, h, l)$, and $\widetilde{\Delta}(F; x, y, h, l)$ by the expression $C_1\Delta(F; x, y, h, l)$ or $\Delta_r^*(F; x, y)$.

Let us now consider double trigonometric series:

$$1)\quad \sum_{m,n=0}^{\infty} \lambda_{m,n} A_{m,n}(x,y), \qquad 2)\quad \sum_{m,n=0}^{\infty} \lambda_{m,n} \frac{\partial}{\partial x} A_{m,n}(x,y),$$

$$3)\quad \sum_{m,n=0}^{\infty} \lambda_{m,n} \frac{\partial}{\partial y} A_{m,n}(x,y), \qquad 4)\quad \sum_{m,n=0}^{\infty} \lambda_{m,n} \frac{\partial^2}{\partial x\, \partial y} A_{m,n}(x,y),$$

$$5)\quad \sum_{m,n=0}^{\infty} \lambda_{m,n} \Delta A_{m,n}(x,y),$$

where

$$\lambda_{m,n} = \begin{cases} \frac{1}{4}, & \text{for } m = n = 0, \\ \frac{1}{2}, & \text{for } m = 0,\ n > 0 \text{ or } n = 0,\ m > 0, \\ 1, & \text{for } m \geq 1,\ n \geq 1, \end{cases}$$

$$A_{m,n}(x, y) = a_{m,n} \cos mx \cos ny + b_{m,n} \sin mx \cos ny +$$

$$+ c_{m,n} \cos mx \sin ny + d_{m,n} \sin mx \sin ny,$$

$$|a_{m,n}| \leq M, \quad |b_{m,n}| \leq M, \quad |c_{m,n}| \leq M, \quad |d_{m,n}| \leq M, \quad M = \text{const}.$$

Series 1) is called $R(\alpha, \beta)\,[R_\lambda(\alpha, \beta)]$-summable ($\alpha$ and β are natural numbers) to the value $S(x, y)$ at the point (x, y), if

$$\lim_{u,v \to 0} R_{\alpha,\beta}(u, v, x, y) = S(x, y) \left[\lim_{(u,v)\lambda \to 0} R_{\alpha,\beta}(u, v, x, y) = S(x, y) \right],$$

where

$$R_{\alpha,\beta}(u, v, x, y) = \sum_{m,n=0}^{\infty} \lambda_{m,n} A_{m,n}(x, y) \left(\frac{\sin mu}{mu} \right)^{\alpha} \left(\frac{\sin nv}{nv} \right)^{\beta}.$$

Starting from series 1), form the series

$$\frac{a_{0,0}}{192}(x^4 + y^4) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{A_{m,0}(x, y)}{m^4} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{A_{0,n}(x, y)}{n^4} + \sum_{m,n=1}^{\infty} \frac{A_{m,n}(x, y)}{(m^2 + n^2)^2},$$

which converges absolutely and uniformly to a certain continuous function $F(x, y)$.

We shall call series 1) (see ([1]), p. 289) R_2^*-summable to the value $S(x, y)$ at the point (x, y), if

$$\Delta^{*(2)} F(x, y) = S(x, y).$$

Theorem 4. Let $f_i(x, y)$, $i = 1, 2, 3, 4$, be arbitrary measurable and finite almost everywhere functions on R. Then there exists such a continuous function $F(x, y)$ that, if 1) is its Fourier series, then almost everywhere on R we have:

1. Series 2) is summable by the method $R_\lambda(2,1)$ to $f_1(x,y)$.
2. Series 3) is summable by the method $R_\lambda(1,2)$ to $f_2(x,y)$.
3. Series 4) is summable by the method $R_\lambda(2,2)$ to $f_3(x,y)$.
4. Series 5) is summable by the method R_2^* to $f_4(x,y)$.

Theorem 5. The Fourier series of any summable function $f(x,y)$ is summable by the method R_2^* almost everywhere on R to the value of this function.

Theorem 6. For every measurable and finite almost everywhere on R function $f(x,y)$, there exists a double trigonometric series 1), summable by the method $R(2,2)$ almost at every point to the value of the function $f(x,y)$.

In the proofs of the theorems presented, various variants of the methods used in the works of N. N. Luzin (see ([2]), pp. 78, 236), I. Marcinkiewicz ([3]), N. K. Bari ([4]), and A. G. Dzhvarsheishvili ([5]) have been applied.

In conclusion, the author considers it his pleasant duty to express gratitude to Prof. A. G. Dzhvarsheishvili for the attention shown to the present work.

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Note: Figure translations are in progress. See original paper for figures.