ON OPERATORS IN HÖLDER SPACES

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Abstract

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MATHEMATICS

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ON OPERATORS IN HÖLDER SPACES

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Everywhere below, $\varphi(\delta)$ $(0 \le \delta \le 1)$ denotes a continuous increasing concave function, $\varphi(0) = 0$. By H_φ we denote the space of functions $u(x)$ continuous on $[0, 1]$, for which the norm

$$\|u\|_\varphi = \max\{\|u\|_C, \sup_{0<\delta\le 1} \omega(u;\delta)/\varphi(\delta)\},$$

is finite, where $\omega(u;\delta)$ is the modulus of continuity of the function $u(x)$. H_φ is called a generalized Hölder space. The classical Hölder spaces H_α are a special case of the spaces H_φ when $\varphi(\delta) = \delta^\alpha$. By H_φ^0 we denote the subspace of those functions from H_φ for which $\omega(u;\delta) = o[\varphi(\delta)]$.

1. Denote by K_φ the cone of nonnegative functions in H_φ. This cone is, obviously, solid and minihedral. It is easy to show that the cone K_φ is not normal (all definitions from the theory of Banach spaces with a cone used here may be found in ([1])).

Theorem 1. *A positive ($AK_\varphi \subset K_{\varphi_1}$) linear (additive and homogeneous) operator A, acting from H_φ into H_{φ_1}, is continuous.*

In the proof of this assertion, the theorem of I. A. Bakhtin, M. A. Krasnosel' skii, and V. Ya. Stetsenko is used (Theorem 2.3 from ([1])). Although the cone K_φ is not normal, the application of this theorem becomes possible thanks to the following simple but useful assertion.

Lemma. *Let a linear operator A defined on a Banach space E_1 act into a Banach space E_2. Let the space E_2 be continuously embedded in a Hausdorff linear topological space $\mathcal{E}$, and suppose that A, as an operator from E_1 into $\mathcal{E}$, is continuous. Then A is continuous as an operator from E_1 into E_2.*

For integral operators in Theorem 1 one can dispense with the positivity condition.

Theorem 2. *If the linear integral operator*

$$Au(x) = \int_0^1 K(x,y)u(y)\,dy \tag{1}$$

acts from H_φ into H_{φ_1}, then it is continuous.

We give some conditions under which a linear integral operator acts from a Banach space E into the Hölder space H_φ. Let M_φ be the Marcinkiewicz space (see, for example, ([2–4])), generated by the concave function $\varphi(\delta)$; M_φ^0 is the subspace of functions from M_φ with absolutely continuous norm.

Theorem 3. *If the function $K(z)$ belongs to M_φ, then the convolution operator*

$$Bu(x) = \int_0^x K(x-y)u(y)\,dy \tag{2}$$

acts into H_φ and is continuous. If $K(z) \in H_\varphi^0$, then this operator is continuous in H_φ^0.

A well-known theorem of L. V. Kantorovich ([5]) on conditions under which the integral operator (1) acts from $L_p(\Omega)$ into $H_\alpha(\Omega)$ admits a generalization to the case of an arbitrary ideal space $E(\Omega)$ ([4]) and the space $H_\varphi(\Omega)$, respectively. For simplicity we restrict ourselves to the formulation of the corresponding assertion in the one-dimensional case. Below, E' denotes the space dual to the ideal space E.

Theorem 4. *Let the kernel $K(x,y)$ of the integral operator (1) be differentiable with respect to x for $x \neq y$. Suppose that the functions $K(x,y)/\varphi(|x-y|)$, $K'_x(x,y)|x-y|/\varphi(|x-y|)$, as functions of y, belong to E' for each x, and*

$$\|K(x,y)/\varphi(|x-y|)\|_{E'} \le M; \qquad \|K'_x(x,y)|x-y|/\varphi(|x-y|)\|_{E'} \le N.$$

Then the operator (1) acts from E into H_φ and is continuous.

2. In the study of operators an important role is played by multiplicative inequalities connecting norms in different spaces (see, for example, ([6],[7])). For the spaces H_α such inequalities were first indicated, apparently, by Kh. Sh. Mukhtarov ([8]). Below we give an analogue of a multiplicative inequality connecting the norm of a function $u(x)$ from H_φ with the norms of this function in the space C and in an arbitrary symmetric space E ([3]). Below, $\psi(\delta)$ denotes the fundamental function of the symmetric space E.

Theorem 5. *The inequality*

$$\|u\|_C \le F(\|u\|_\varphi, \|u\|_E) \qquad (u(x) \in H_\varphi), \tag{3}$$

holds, where

$$F(t,s)=\min_{0<\delta\le 1/2}\left[t\varphi(\delta)+\frac{s}{\psi(\delta)}\right]\qquad (0\le s\le t\psi(1)).$$

The function $F(t,s)$ has the following properties:

1) $F(t,s)$ is jointly continuous in the variables and is a nondecreasing concave function in each of the variables.

2) $F(t,0)=0$ for all $t\ge 0$.

3) $F(t,kt)=c(k)t$.

For the case where $\psi(\delta)=[\varphi(\delta)]^r$ $(r>0)$, inequality (3) has the form

$$\|u\|_C\le c(r)\|u\|_\varphi^{r/(1+r)}\|u\|_E^{1/(1+r)}\qquad (u(x)\in H_\varphi).$$

In particular, for $\psi(\delta)=\varphi(\delta)$,

$$\|u\|_C\le c\sqrt{\|u\|_\varphi\|u\|_E}.$$

It follows from inequality (3), for example, that every sequence of functions bounded in H_φ and converging in measure converges uniformly.

3. Denote by f the superposition operator generated by a certain function $g(x,u)$:

$$fu(x)=g[x,u(x)]. \tag{4}$$

As an operator acting in various spaces of summable functions, it has been studied by many authors (see, for example, ([4],[6]), where a bibliography is given). As an operator acting in Hölder spaces, the operator (4) has in general been studied hardly at all. Some results for the simplest case, when the function $g(x,u)\equiv g(u)$ does not depend on x, were obtained by A. A. Babaev (for the spaces H_α) and Kh. Sh. Mukhtarov (for the spaces H_φ). A more complete investigation of such a simplest operator was carried out by one of the authors of the present note in ([9]). Here we formulate two theorems on the operator (4) in the general case.

Denote by $\mathfrak{M}$ the class of increasing functions $a(t)$, continuous on $[0,\infty)$, satisfying the conditions:

a) $\alpha(t)>0$ for $t>0$, $\alpha(0)=0$;

b) $\lim\limits_{\delta\to 0}\alpha[\varphi(\delta)]/\varphi_1(\delta)>0$.

By $\mathfrak{M}_1$ we shall denote the class of such functions $\alpha(t)$ which, instead of condition b), satisfy the condition

b′) $\lim_{\delta\to 0} \alpha[l\varphi(\delta)]/\varphi_1(\delta) < \infty$ for every $l > 0$.

Theorem 6. In order that the operator (4) act from H_φ to H_{φ_1} and be bounded, it is necessary that, for every $r > 0$ and every function $\alpha(t)$ from $\mathfrak{M}$, the function $g(x, u)$ satisfy the condition

$$|g(x_1, u_1) - g(x_2, u_2)| \leq M(r, \alpha)\, [\varphi_1(|x_1 - x_2|) + \alpha(|u_1 - u_2|)]$$

$$(x_1, x_2 \in [0, 1], \quad u_1, u_2 \in [-r, r]), \tag{5}$$

where the constant $M(r, \alpha)$ does not depend on x, u.

If, for some function $\alpha(t)$ from $\mathfrak{M}_1$, (5) is satisfied, then the operator (4) acts from H_φ to H_{φ_1} and is bounded.

A consequence of this theorem is

Theorem 7. In order that the operator (4) act in H_φ and be bounded, it is necessary and sufficient that, for every $r > 0$, the function $g(x, u)$ satisfy the condition

$$|g(x_1, u_1) - g(x_2, u_2)| \leq M(r)\, [\varphi(|x_1 - x_2|) + |u_1 - u_2|]$$

$$(x_1, x_2 \in [0, 1], \quad u_1, u_2 \in [-r, r]).$$

The sufficiency of the conditions of these theorems is obvious. It has been used by many authors.

In [9], for the operator (4) generated by a function $g(u)$ independent of x, analogous theorems were proved without the additional assumption of boundedness of the operator f. In the general case this assumption is essential, since one can construct unbounded operators (4) acting in H_φ.

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Note: Figure translations are in progress. See original paper for figures.