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ON SUBSPACES OF A COUNTABLE INDUCTIVE LIMIT

MATHEMATICS

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Abstract

Full Text

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MATHEMATICS

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ON SUBSPACES OF A COUNTABLE INDUCTIVE LIMIT

(Presented by Academician A. N. Kolmogorov, 20 III 1970)

Let $E = \lim_{\rightarrow} E_n$ be an inductive limit of locally convex spaces and let H be a subspace of E (in the topology induced from E). We shall say that H has property (T_0) if $H' = (\lim_{\rightarrow}(E_n \cap H))'$, and property (T) if $H = \lim_{\rightarrow}(E_n \cap H)$. Obviously, (T) implies (T_0). We shall give necessary and sufficient conditions for the fulfillment of properties (T_0) and (T) under certain assumptions concerning E.

1. Let $G = \lim_{\rightarrow} G_n$ be an inductive limit of locally convex spaces. We shall say that G has property (M_0) if all G_n are of countable type* and contain an absolutely convex neighborhood of zero U_n such that $U_n \subset U_{n+1}$ and

$$\forall i\ \exists j:\ \forall k > j\ \forall f \in G_j'\ \forall \varepsilon > 0\ \exists g \in G_k':\ |f(x) - g(x)| < \varepsilon\ \forall x \in U_i. \qquad (*)$$

If condition $(*)$ is replaced by the following: $\forall i\ \exists j:\ \ \forall k > j$ the topologies induced on U_i from G_j and G_k coincide, then we shall say that G has property (M). (M) implies (M_0). If $E = \lim_{\rightarrow} E_n$ is a strict inductive limit of spaces of countable type, then E has property (M).

Theorem 1. If

$$E/H = \lim_{\rightarrow} E_n/E_n \cap H$$

has property (M_0), then H has property (T_0). If E has property (M_0), or E is a strict inductive limit** and all $E_n/E_n \cap H$ are of countable type, then the converse assertion is also true.

Theorem 2. If E/H has property (M), then H has property (T). If E has property (M), or E is a strict inductive limit** and all $E_n/E_n \cap H$ are of countable type, then the converse assertion is also true.

Remark. According to ([7]), if E is a strict inductive limit of Fréchet-Schwartz spaces, then for a closed H property (T_0) is equivalent to property (T).

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For the proof of Theorems 1 and 2 one must use the scheme of the proofs of Theorem 2 from ([3]) and Theorem 1′ from ([4]), and Theorem 3 below.

We shall give the example, constructed by V. P. Palamodov and the author, of a closed subspace H of the space $D((-1,2))$ not having property (T_0). It differs from the example in ([6]) by its simplicity and by the fact that H is the image of an operator from the space D.

Let $\varphi \in D(R)$, with $\operatorname{supp}\varphi = [1,2]$. Define a mapping

$$Q : D((-1,0)) \to D((-1,2))$$

by the formula $Q(f) = f + f * \varphi$. Put $H = \operatorname{Im} Q$,

$$E_n = D\left([-1+2^{-n},\, 2-2^{-n}]\right), \qquad H_n = E_n \cap H.$$

It is easy to see that Q is injective and H is closed. Suppose that H has property (T_0). Since E is a strict–

* I.e., they possess a countable fundamental system of neighborhoods of zero.
** In this case it is not assumed that all E_n are of countable type.

inductive limit, then by Theorem 1 there exist a neighborhood of zero V_3 in E_3 and $j > 3$ such that

$$\forall u \in E'_j, \quad u|_{H_j} = 0 \quad \forall \varepsilon > 0\, \exists v \in E'_{j+1}, \quad v|_{H_{j+1}} = 0 : \ |u(x) - v(x)| < \varepsilon \ \forall x \in V_3.$$

Obviously $\delta_{-2^{-j}}|_{H_j} = 0$. Let $v \in E'_{j+1}$ and $v|_{H_{j+1}} = 0$. The latter means that $v(f + f*\varphi) = 0$, $\forall f \in D([-1+2^{-j-1}, -2^{-j-1}])$, whence it follows that v is infinitely differentiable on $[-1+2^{-j-1}, 2^{-j-1}]$. But $\delta_{-2^{-j}}$ cannot be approximated by infinitely differentiable functions on V_3. We have arrived at a contradiction; consequently, H does not have property (T_0).

Let us note that it follows from this that Q^{-1} is discontinuous.

From Theorem 2, with the aid of the 3×3-lemma for the category of locally convex spaces ([2]), there follows the main result of ([5]). However, instead of Theorem 2 it suffices to use the weaker Theorem 1 from ([3]).

II. Let L be the category of vector spaces over the field $\mathbf{C}$ or $\mathbf{R}$. By $\widetilde{L}$ we shall denote the category of inverse spectra over L with the natural series $\mathbf{N}$, ordered increasingly, as the common set of indices.

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V. P. Palamodov in [1] studied the derived functors of the left exact functor $\operatorname{Pro}: \widetilde{L} \to L$, which assigns to each inverse spectrum $\mathcal{X} = \{X_p, \alpha_p^q : X_q \to X_p\}$ from $\widetilde{L}$ its projective limit. He found a necessary and sufficient condition for $\operatorname{Pro}^1 \mathcal{X}$ to vanish in the case when the spaces X_p ($p \in \mathbf{N}$) can be endowed with the topologies of Fréchet spaces so that the mappings α_p^q are continuous.

We shall prove an analogous theorem for the dual, in a certain sense, case.

Definition. An absolutely convex set B in a vector space E will be called a Banach disk if the space E_B, spanned by B and endowed, as a seminorm, with its Minkowski functional, is Banach.

Theorem 3. *Let in each space X_p of the inverse spectrum*

$$\mathcal{X} = \{X_p, \alpha_p^q\}$$

from $\widetilde{L}$ there exist a sequence of Banach disks

$$(B_p^k)_{k\in\mathbf{N}}$$

such that

$$\bigcup_{k=1}^{\infty} B_p^k = X_p$$

and

$$[(\alpha_p^{p+1})^{-1}(B_p^k)] \cap B_{p+1}^r$$

is a Banach disk for arbitrary p, k, r. Then $\operatorname{Pro}^1 \mathcal{X} = 0$ if and only if in each X_p there exists a Banach disk B_p such that

$$\alpha_p^{p+1}(B_{p+1}) \subset B_p$$

and

$$\forall i \ \exists j : \ \alpha_i^j(X_j) \subset B_i + \alpha_i(\operatorname{Pro} \mathcal{X}).$$

For the proof we shall need a lemma which in essence belongs to Banach.

Lemma. *Let X, Y be vector spaces and $f : X \to Y$ a linear mapping. Suppose that the sequences of absolutely convex sets $(U_n)_{n\in\mathbf{N}}$ and $(V_n)_{n\in\mathbf{N}}$ in X and Y, respectively, form fundamental systems of neighborhoods of zero for the topologies of separable additive groups. If the group X is complete, the graph of f is closed, and $f(U_n)$ is, for every n, of second category in Y, then f is open.*

Proof of Theorem 3. Sufficiency follows from Theorem 11.2 in [1]. We shall prove necessity.

Put

$$\Pi = \prod_{p=1}^{\infty} X_p.$$

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According to ([1]), if $\operatorname{Pro}^1 \mathcal{X} = 0$, then the mapping

$$\pi : \Pi \to \Pi$$

defined by the formula

$$\pi(x_1, x_2, x_3, ...) = (x_1 - \alpha_1^2(x_2),\ x_2 - \alpha_2^3(x_3), ...) \qquad (**)$$

is epimorphic. Introduce in Π the topology φ of an additive group with a fundamental system of neighborhoods of zero $(V_n)_{n\in\mathbf{N}}$, where

$$V_n = 0 \times \cdots \times 0 \times X_n \times X_{n+1} \times \cdots$$

(the 0 occurring $n-1$ times). Since (Π, φ) is complete and π is epimorphic, for each p -

there exists B_n^k such that $\pi(B_1^{k_1} \times ... \times B_m^{k_m} \times X_{m+1} \times X_{m+2} \times ...)$ is of the second category in (Π, φ), for every m. Introduce on Π the topology τ of a group with a fundamental system of neighborhoods of zero $(U_n)_{n\in\mathbb{N}}$, where

$$U_n = 2^{-n} B_1^{k_1} \times ... \times 2^{-n} B_n^{k_n} \times X_{n+1} \times X_{n+2} \times ... .$$

By the lemma, $\pi : (\Pi, \tau) \to (\Pi, \varphi)$ is open. Hence $\forall i\ \exists j :$

$$\pi(B_1^{k_1} \times ... \times B_i^{k_i} \times X_{i+1} \times X_{i+2} \times ...) \subset (0 \times ... 0 \times X_j \times 0 \times ...).$$

To complete the proof one must use formula (**).

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