ON A GENERALIZATION OF CRAIG' S THEOREM

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Abstract

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ON A GENERALIZATION OF CRAIG' S THEOREM

(Presented by Academician Yu. V. Linnik on June 8, 1970)

Let $X = (x_1, \dots, x_n)$ be an n-dimensional vector with independent components having the normalized normal distribution, and let $Q_1(x_1, \dots, x_n)$ and $Q_2(x_1, \dots, x_n)$ be two quadratic forms

$$Q_1 = X'AX, \qquad Q_2 = X'BX.$$

Craig' s theorem is well known (see ([1])): in order that the quadratic forms Q_1 and Q_2 be independent, it is necessary and sufficient that

$$A \cdot B = 0. \tag{1}$$

Let us formulate this result differently.

By an orthogonal transformation taking X into a vector y with independent and normalized normal components, Q_1 can be transformed to the form

$$Q_1 = \sum_{i=1}^{r} a_i y_i^2, \qquad r \leq n. \tag{2}$$

Craig' s result means that Q_1 and Q_2 are independent if and only if Q_2 depends only on $y_{r+1}, \dots, y_n$.

We have proved the following generalization of Craig' s result.

Theorem. Let $X = (x_1, \dots, x_n)$ be an n-dimensional vector with independent components distributed normally with mean 0 and variance 1. Let $Q(x_1, \dots, x_n)$ be a quadratic polynomial in $x_1, \dots, x_n$, and let $P(x_1, \dots, x_n)$ be an arbitrary polynomial statistic. For the independence of P and Q it is necessary and sufficient that there exist an orthogonal transformation A

$$Ax = y,$$

such that Q depends only on $y_1, \dots, y_r$, while P depends only on $y_{r+1}, \dots, y_n$.

In other words: if the polynomial Q is brought by an orthogonal transformation to the form

$$Q = \sum_{j=1}^{s} \lambda_j y_j^2 + \sum_{i=q}^{p} \mu_i y_i,$$

$$q \leq s+1, \qquad \lambda_i \neq 0, \qquad \mu_i \neq 0, \qquad \max(s,p) \neq r \leq n,$$

then the polynomials P and Q are independent if and only if P depends only on $y_{r+1}, \dots, y_n$.

Our result will essentially not change if it is assumed that X has a multivariate nondegenerate normal distribution with mean 0 and covariance matrix V.

Indeed, the matrix V can be represented in the form

$$V = T \cdot T',$$

where T is a real matrix, and the transformation $X = Ty$ carries the exponent of the exponential corresponding to the distribution density of X from $X'V^{-1}X$ into $y'T'V^{-1}Ty = y'y$. Thus the general case reduces to the one already studied.

A. A. Zinger informed the author that he had proved an analogous result for the case of a quadratic form Q.

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REFERENCES

1. A. T. Craig, *Ann. Math. Statist.*, **14**, 195 (1943).

Note: Figure translations are in progress. See original paper for figures.